

A REVERSE ISOPERIMETRIC INEQUALITY FOR J-HOLOMORPHIC CURVES

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ABSTRACT. We prove that the length of the boundary of a J -holomorphic curve with Lagrangian boundary conditions is dominated by a constant times its area. The constant depends on the symplectic form, the almost complex structure, the Lagrangian boundary conditions and the genus. A similar result holds for the length of the real part of a real J -holomorphic curve. The infimum over J of the constant properly normalized gives an invariant of Lagrangian submanifolds. We calculate this invariant to be 2π for the Lagrangian submanifold $\mathbb{R}P^n \subset \mathbb{C}P^n$. Furthermore, we apply our result to prove compactness of moduli of J -holomorphic maps to non-compact target spaces that are asymptotically exact.

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1. INTRODUCTION

1.1. A consequence of the Cauchy-Crofton formula. We begin with a bound on the length of a real algebraic curve in terms of its degree, which we learned from [8]. Let γ be a one dimensional submanifold of \mathbb{RP}^n . Let h be the round metric on \mathbb{RP}^n normalized so the length of a line is 1. Let dV be the volume form on $Gr(n, n+1)$, the Grassmanian of hyperplanes in \mathbb{R}^{n+1} , that is invariant under the induced action of the isometry group of h and satisfies

$$\int_{Gr(n, n+1)} dV = 1.$$

For $H \in Gr(n, n+1)$ denote by $N(H)$ the number of intersection points between H and γ ,

$$N(H) = N(H, \gamma) = |\{\gamma \cap H\}|.$$

By transversality, $N(H)$ is finite for generic H . Let $\ell(\gamma; h)$ denote the length of γ with respect to h . The Cauchy-Crofton formula [20, eq. 12] asserts that

$$(1) \quad \int_{H \in Gr(n, n+1)} N(H) dV = \ell(\gamma; h).$$

For a quick sanity check, note that when γ is a straight line, both sides of equation (1) are equal to 1.

Assume now that γ is an algebraic curve of degree d . As observed in [8, p.45], the Cauchy-Crofton formula (1) implies

$$(2) \quad d \geq \ell(\gamma; h).$$

Inequality (2) can be interpreted as a reverse isoperimetric inequality. Indeed, let $(\Sigma, \partial\Sigma)$ be a Riemann surface with boundary and let

$$u : (\Sigma, \partial\Sigma) \rightarrow (\mathbb{CP}^n, \mathbb{RP}^n)$$

be a holomorphic map. The Fubini-Study metric on $\mathbb{C}P^n$ normalized so the area of a complex line is $\frac{1}{\pi}$ induces the metric h on $\mathbb{R}P^n$, so we denote it also by h . By Wirtinger's theorem,

$$Area(u; h) = \frac{d}{2\pi}.$$

Schwarz reflection implies that $u|_{\partial\Sigma}$ is real algebraic. So equation (2) gives the estimate

$$(3) \quad 2\pi Area(u; h) \geq \ell(u|_{\partial\Sigma}; h).$$

The present paper extends inequality (3) to general symplectic manifolds M , Lagrangian submanifolds $L \subset M$, and J -holomorphic curves with boundary in L .

1.2. The compact case. In the following let (M, ω) be a symplectic manifold, let $L \subset M$ be a Lagrangian submanifold and let J be an almost complex structure on M for which the form $\omega(\cdot, J\cdot)$ is positive definite. Denote by g_J the symmetrization of the form $\omega(\cdot, J\cdot)$. For Σ a Riemann surface with boundary we denote by $\Sigma_{\mathbb{C}}$ the complex double of Σ .

Theorem 1.1. *Suppose M is compact. There are constants $f_1 = f_1(J, \omega, L)$ and $g_1 = g_1(J, \omega, L)$, homogeneous of degrees $-\frac{1}{2}$ and $\frac{1}{2}$ respectively in ω , with the following significance. For any Riemann surface Σ with boundary, and for any J -holomorphic curve*

$$u : (\Sigma, \partial\Sigma) \rightarrow (M, L),$$

we have

$$(4) \quad \ell(u|_{\partial\Sigma}; g_J) \leq f_1 Area(u; g_J) + g_1 genus(\Sigma_{\mathbb{C}}).$$

Let $(M, L) = (\mathbb{C}P^n, \mathbb{R}P^n)$, let $\omega = \omega_{FS}$ be the Fubini Study form and let $J = J_{st}$ be the standard complex structure. By the discussion in Section 1.1, we may take

$$f_1(J_{st}, \omega_{FS}, L) = 2\pi, \quad g_1(J_{st}, \omega_{FS}, L) = 0.$$

At this point it is not clear whether the genus dependence in (4) can be eliminated in the general case. On the other hand, the monotonicity inequality [21] gives a lower bound on $Area(u; g_J)$. So the constant g_1 can be set to zero at the expense of making f_1 genus dependent.

Recently, real symplectic geometry has attracted considerable attention. See, for example, [23, 19, 11, 4]. Theorem 1.1 has the following parallel in the real-symplectic setting. Recall that a **real** symplectic manifold is a pair (M, ϕ) where M is a symplectic manifold and

$\phi : M \rightarrow M$ is an anti-symplectic involution. The natural compatibility condition for almost complex structures is that $\phi^*J = -J$. A **real** Riemann surface is a pair (Σ, ψ) , where Σ is a Riemann surface and $\psi : \Sigma \rightarrow \Sigma$ is an anti-holomorphic involution. We denote by $\Sigma_{\mathbb{R}}$ the fixed point set of ψ . A J -holomorphic curve $u : \Sigma \rightarrow M$ is called **real** if $\phi \circ u = u \circ \psi$.

Theorem 1.2. *Suppose (M, ϕ) is a compact real symplectic manifold. There are constants $f_2 = f_2(J, \omega)$ and $g_2 = g_2(J, \omega)$, homogeneous of degrees $-\frac{1}{2}$ and $\frac{1}{2}$ respectively in ω , with the following significance. For any real Riemann surface (Σ, ψ) and any real J -holomorphic curve $u : \Sigma \rightarrow M$, we have*

$$(5) \quad \ell(u|_{\Sigma_{\mathbb{R}}}; g_J) \leq f_2 \text{Area}(u; g_J) + g_2 \text{genus}(\Sigma).$$

We remark that a forward isoperimetric inequality does not hold for holomorphic curves. Indeed, consider degree 2 curves in $(\mathbb{CP}^2, \mathbb{RP}^2)$. For $t > 0$, let C_t be the closure of one of the two connected components of the non-real solutions of the equation $X^2 + Y^2 - t = 0$. Then C_t has constant area but vanishing boundary length as t goes to 0.

1.3. The optimal isoperimetric constant. The preceding theorems, though they involve Riemannian length measurements, lead to a purely symplectic invariant of Lagrangian submanifolds. For a given J , denote by $F_1(\omega, J, L)$ the optimal value of the constant $f_1(\omega, J, L)$ of Theorem 1.1 when u ranges over J -holomorphic maps from the surface of genus 0. Define the constant $h_1(M, L, \omega)$ by

$$h_1(M, L, \omega) = \inf_J \frac{F_1(\omega, J, L)}{2 \text{Diam}(L; g_J)},$$

where the infimum is over all J tamed by ω . Similarly, for (M, ϕ) a real symplectic manifold, for given ϕ anti-invariant J , denote by $F_2(\omega, J)$ the optimal value of the constant $f_2(\omega, J)$ of Theorem 1.2 when u ranges over maps from the surface of genus 0. Define the constant $h_2(M, \phi, \omega)$ by

$$h_2(M, \phi, \omega) = \inf_J \frac{F_2(\omega, J)}{2 \text{Diam}(\text{Fix}(\phi); g_J)},$$

where the infimum is over all tame J such that $\phi^*J = -J$.

Clearly, h_1 and h_2 are symplectic invariants. Theorems 1.1 and 1.2 imply that $h_i(M, L, \omega) < \infty$. As seen in the proof of the following proposition, lower bounds follow from open Gromov-Witten theory.

Proposition 1.3. *Let $M = \mathbb{CP}^n$, let $L = \mathbb{RP}^n$ and let $\omega = \omega_{FS}$ be the Fubini Study form normalized so the area of a line is $\frac{1}{\pi}$. Then $h_1(\omega) = 2\pi$.*

For lower bounds on h_2 , we can use the rapidly developing theory of Welschinger invariants [23, 11, 4, 10]. In the projective real algebraic case, the discussion of Section 1.1 gives upper bounds on h_2 . However, to minimize the discrepancy between upper and lower bounds, it is necessary to understand how to maximize diameter within a deformation class of projective real algebraic varieties.

There are various ways to generalize h_i to higher genus, and it seems interesting to study the resulting invariants. Also, restricting to u with non-trivial boundary degree, it could be interesting to compare the isoperimetric constant with the 1-systole of L . See [12] for background on systolic geometry. We leave these problems for future research.

1.4. The general case. We show how Theorems 1.1 and 1.2 generalize when M and L are not compact. In the process, we characterize more precisely the dependence of the isoperimetric constants f_i on the geometry of (M, ω, J, L) . Let $K > 1$. We say the quadruple (M, ω, J, L) has **K -bounded** geometry under the following conditions: With respect to the metric g_J , the curvature of M , the almost complex structure J , the second fundamental form of L , and derivatives thereof, are bounded from above by K . Moreover, the injectivity radii of M and L are bounded from below by $\frac{1}{K}$. We say that a submanifold P of a Riemannian manifold Q has a tubular neighborhood of **width** $1/K$ in the following situation: Denoting by N_P the normal bundle of P and by O the zero section of N_P , we have that $\exp|_{B_{1/K}(O)}$ is a diffeomorphism onto its image.

Theorem 1.4. *There are functions $f_1 = f_1(K)$ and $g_1 = g_1(K)$, with the following significance. Theorem 1.1 holds upon replacing the assumption that M is compact by the assumption that (M, ω, L, J) has K -bounded geometry as well as one of the following:*

- (a) *L has a tubular neighborhood of width $\frac{1}{K}$.*
- (b) *Fix a conformal metric of constant curvature and unit volume on Σ such that $\partial\Sigma$ is totally geodesic. Then $\partial\Sigma$ and each connected component of L have tubular neighborhoods of width $\frac{1}{K}$.*

Theorem 1.2 generalizes to the non-compact case under considerably weaker assumptions.

Theorem 1.5. *There are functions $f_2(K)$ and $g_2(K)$ such that Theorem 1.2 holds upon replacing the requirement that M be compact by the following: The curvature of g_J and the derivatives of J are bounded from above by K , whereas the radius of injectivity is bounded from below by $\frac{1}{K}$.*

There is also an a priori estimate on the diameter of J -holomorphic curves.

Theorem 1.6. *There are functions $f_3 = f_3(K)$ and $g_3 = g_3(K)$ such that under the same assumptions as in Theorem 1.4 and denoting $b := |\pi_0(\partial\Sigma)|$,*

$$(6) \quad \text{Diam}(u(\Sigma); g_J) \leq (b+1)[f_3 \text{Area}(u; g_J) + g_3 \text{genus}(\Sigma_{\mathbb{C}})].$$

In the case of closed curves, which includes conjugation invariant curves, as well as in case (a) of Theorem 1.4, the diameter estimate (6), without any genus dependence, was proved by Sikorav [21] using the monotonicity inequality. However, Sikorav's technique is image oriented, so we could not see how it would allow one to utilize the bounds on the domain necessary in case (b) of Theorem 1.4. More importantly, we could not see how to generalize Sikorav's technique to obtain results on boundary length.

1.5. An example. The following example, due to [16], illustrates the role of conditions (a) and (b) of Theorem 1.4. Consider the special Lagrangian fibration of $M = \mathbb{C}^3$ discussed in [9] and [1]. Namely, let $H : \mathbb{C}^3 \rightarrow \mathbb{R}^3$, be given by

$$(z_1, z_2, z_3) \mapsto (|z_1|^2 - |z_3|^2, |z_2|^2 - |z_3|^2, \text{Im}(z_1 z_2 z_3)).$$

It can be shown that for each $c \in \mathbb{C}^3$ the fiber $H^{-1}(c)$ is Lagrangian. Moreover, letting J_0 and ω_0 denote the standard complex and symplectic structures on \mathbb{C}^3 , it can be shown that $(M, \omega_0, J_0, H^{-1}(c))$ has uniformly bounded geometry and that $H^{-1}(c)$ has a tubular neighborhood of uniform width. Thus the reverse isoperimetric inequality applies to curves with boundary in $H^{-1}(c)$.

On the other hand, consider a Lagrangian $L \subset \mathbb{C}^3$ which is the union of two or more fibers of H . Then it is easily shown that the components of L are arbitrarily close to each other at infinity. Thus L as a whole does not have a tubular neighborhood of uniform width, but each component of L does have such a tubular neighborhood. We construct a counterexample to the reverse isoperimetric inequality as follows. Let $L_0 = H^{-1}(0, 0, 0)$ and $L_1 = H^{-1}(-1, -1, 0)$. For any $a \in \mathbb{R}$ we construct a holomorphic annulus with one boundary component in L_0 and the other one in L_1 . For any $a \in \mathbb{R}$ let r_a be a positive solution of the equation

$$a^2 r^6 + r^4 - a^2 = 0.$$

Let Σ_a be the annulus in the plane with radii 1 and r_a . Consider the map

$$(\Sigma_a, \partial\Sigma_a) \rightarrow (\mathbb{C}^3, L_0 \cup L_1)$$

given by

$$z \mapsto \left(az, az, \frac{a}{z^2} \right).$$

Allowing a to approach infinity, the boundary length is unbounded while the area, being a homological invariant, remains constant. Thus there is no reverse isoperimetric inequality in this case. So, Theorem 1.4(b) implies that the width of any tubular neighborhood of $\partial\Sigma_a$ goes to 0 as a approaches infinity. This can indeed be verified directly by noting that $\lim_{a \rightarrow \infty} r_a = 1$, so that $\text{Mod}(\Sigma_a) = \ln r_a \rightarrow 0$.

1.6. Application. We apply Theorem 1.4 to deduce compactness of moduli of J -holomorphic maps in the following scenario. Our argument can be seen as a quantitative version of the idea employed in [13]. We say that ω is **asymptotically exact** if for a point $p \in M$ there exists a 1-form λ such that

$$\lim_{R \rightarrow \infty} \|(\omega - d\lambda)|_{M \setminus B_R(p)}\| = 0.$$

Similarly, we say that L is an asymptotically exact Lagrangian submanifold if ω is asymptotically exact and there is a function $f : L \rightarrow \mathbb{R}$ so that

$$\lim_{R \rightarrow \infty} \|(\lambda - df)|_{L \setminus B_R(p)}\| = 0.$$

Applying Stokes theorem, the following is immediate.

Corollary 1.7. *Let $A \in H_2(M, L)$, $p \in M$, $K > 1$, and assume ω and L are asymptotically exact. Then there is an $R = R(M, L, p, K)$ such that any J -holomorphic*

$$u : (\Sigma, \partial\Sigma) \rightarrow (M, L)$$

with $[u] = A$ that satisfies the conditions of Theorem 1.4 also satisfies $u(\Sigma) \subset B_R(p)$.

Given this corollary, standard Gromov compactness implies compactness of moduli. In a paper to appear subsequently, we prove that toric Calabi Yau manifolds along with the Lagrangian submanifolds of Aganagic Vafa [2] are asymptotically exact and have bounded geometry. Thus, we apply the construction of [17] to define open Gromov-Witten invariants for general toric Calabi Yau 3-folds.

1.7. Idea of the proof. We restrict attention to the case of real J -holomorphic maps from a real Riemann surface (Σ, ψ) , which for the time being, we assume to be a sphere. We assume the fixed point set $\Sigma_{\mathbb{R}}$ of ψ is non-empty and abbreviate $\gamma = \Sigma_{\mathbb{R}}$. We equip Σ with a round metric invariant under ψ of radius 1. In particular, γ is a great circle.

Let $u : (\Sigma, \gamma) \rightarrow (M, L)$ be real J -holomorphic. Let $f : \gamma \rightarrow [0, \infty)$ be given by $f(x) := \|du(x)\|$. Then $\ell(u(\gamma))$ is the area of the hypograph of f ,

$$H = \{(x, t) \in \gamma \times [0, \infty) | t \leq f(x)\}.$$

The bubbling phenomenon implies that there is no a priori bound on the L_∞ norm of f in terms of the energy of u . Thus our argument relies on a close analysis of the set H .

In the following sketch of our proof, we make the simplifying assumption that f has a unique local maximum on γ . For $t \in [1, \infty)$ write $w_t = \{x \in \gamma | f(x) \geq t\}$ and let $W_t \subset \Sigma$ be the image under the exponential map of the ball of radius $\frac{1}{t}$ in the normal bundle of w_t . Our simplifying assumption implies that w_t is connected. Note also that if $t' > t$, then $w_{t'} \subset w_t$. For any t write

$$a_t := \left\lfloor \frac{t\ell(w_t)}{2} \right\rfloor.$$

One the main ways the fact that u is J -holomorphic enters our proof is the following energy quantization property. Let $p \in \Sigma$ with $d = \|du(p)\| \geq 1$, and let $B \subset \Sigma$ be the disk of radius $\frac{1}{d}$ centered at p . Then it is known that

$$(7) \quad \int_B \|du\|^2 \geq \delta,$$

where δ is a constant that depends only on the geometry of (M, ω, J, L) . From now on, we call a disk $B \subset \Sigma$ satisfying (7) a **dense disk**.

Let $R_i = w_{3^i} \times [0, 3^{i+1}]$ for $0 \leq i \leq \log_3 \sup_{\partial\Sigma} f$. Clearly, the rectangles R_i cover the area under the graph of f wherever $f \geq 1$. Thus to deduce Theorem 1.2, it suffices to bound the sum

$$S := \sum_i 3^{i+1} \ell(w_{3^i}).$$

To get such a bound, note that for any $i \in \mathbb{N}$, by the energy quantization property, the set $W_{3^i} \setminus W_{3^{i+1}}$ contains at least n_i disjoint dense disks, where

$$n_i := a_{3^i} - \left\lfloor \frac{1}{3}(a_{3^{i+1}} - 1) \right\rfloor - 2.$$

See Figure 1. The -2 in the formula expresses the fact that an arbitrarily small neighborhood of each end point of $W_{3^{i+1}}$ can knock out a whole dense disk of W_{3^i} . Since area coincides with energy for J -holomorphic maps, estimate (7) implies the bound

$$\sum n_i \leq \frac{\text{Area}(\Sigma; g_J)}{\delta}.$$

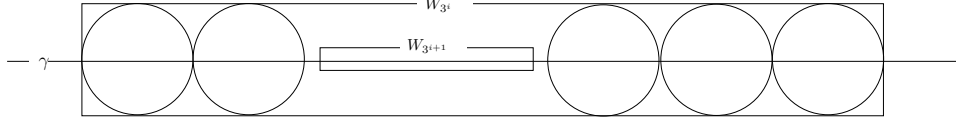


FIGURE 1.

Assume momentarily that for each i we have

$$(8) \quad a_{3^i} \geq 4.$$

Then

$$a_{3^{i+1}} - \left\lfloor \frac{1}{3}(a_{3^{i+1}} - 1) \right\rfloor - 2 \geq \frac{1}{4}a_{3^{i+1}}.$$

So,

$$\sum n_i \geq \frac{1}{4} \sum a_{3^i} \geq \frac{1}{4} \sum \left\lfloor \frac{1}{2} 3^i \ell(w_{3^i}) \right\rfloor.$$

But, again using assumption (8), we have

$$\left\lfloor \frac{1}{2} 3^i \ell(w_{3^i}) \right\rfloor \geq \frac{4}{5} \cdot \frac{1}{2} 3^i \ell(w_{3^i}).$$

So,

$$\frac{\text{Area}(\Sigma; g_J)}{\delta} \geq \sum n_i \geq \frac{1}{4} \cdot \frac{4}{5} \cdot \frac{1}{2} \sum 3^i \ell(w_{3^i}) \geq \frac{1}{5 \cdot 6} S,$$

giving the required bound. The argument breaks down, however, if we remove assumption (8) as in Figure 2.

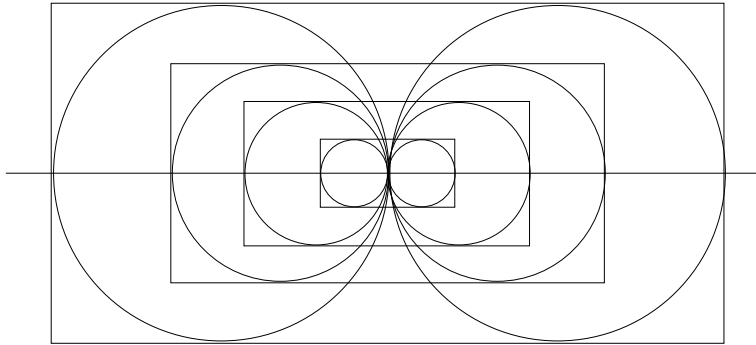


FIGURE 2.

To deal with this problem, we partition H into the sets

$$K = \left\{ (x, t) \in H \left| \ell(w_t) \geq \frac{8}{t}, t \geq 1 \right. \right\},$$

and $N = H \setminus K$. These are the thick and thin parts of the hypograph respectively. Figure 3 shows a possible alternation between thick and thin. The solid line is the graph of f , and the dashed line is the graph of the function

$$x \mapsto \frac{8}{d(x, x_0)},$$

where x_0 is the point where f obtains its maximum. Conceptually, the components of the thick part should be thought of as parts of γ which lie on bubbles of u , and those of the thin part as the necks separating the bubbles. A slight modification of the above argument shows the same linear lower bound on the area of the thick part. For the thin part we utilize the bound on the width of the graph and well known properties of holomorphic annuli.

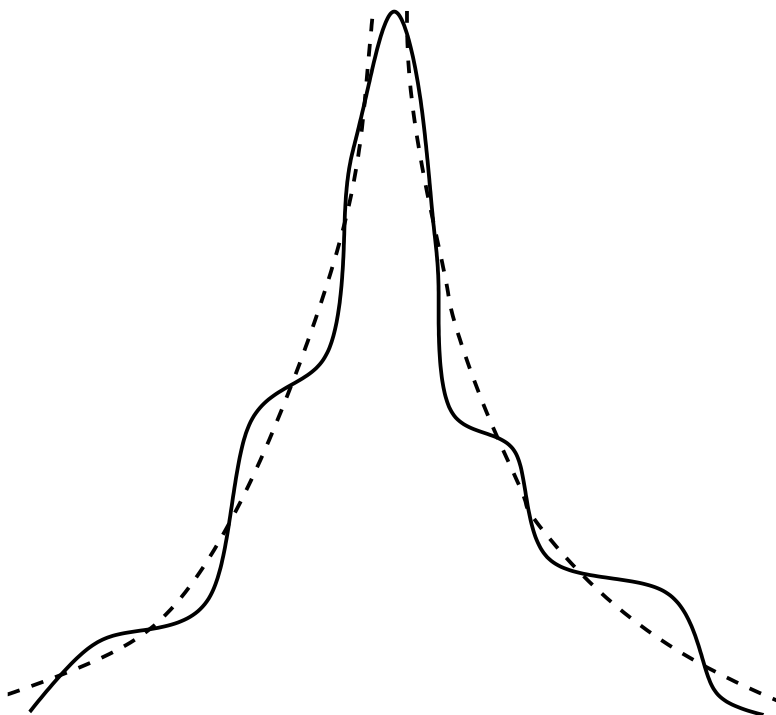


FIGURE 3.

When we consider maps from surfaces with genus greater than 0, we also have to deal with the fact that the domain has unbounded length. For values of the derivative which are small relative to the radius of injectivity, the above argument again breaks down. Integrating a small number over an unbounded domain gives an unbounded value. This is again dealt with by utilizing the properties of holomorphic annuli. Our

proof thus exhibits a pleasing symmetry between the thin part of the graph and the thin part of the surface.

The main technical difficulties in the proof arise from the a priori arbitrary arrangement of the critical points of f . A large part of the proof is devoted to constructing a partition of the hypograph of f into a thick part and a thin part in such away that the above arguments apply.

The paper is organized as follows. Section 2 reviews basic notions of the conformal geometry of surfaces as well as the thick thin decomposition for Riemann surfaces with negative Euler characteristic equipped with a hyperbolic metric. Section 3 presents the concept of thick thin measures. The measure one should have in mind is the energy of a J -holomorphic map from the surface. Section 4 formulates Theorem 4.2, which is a generalization of the theorems in this introduction. It addresses arbitrary conjugation invariant geodesics in the complex double $\Sigma_{\mathbb{C}}$. Section 5 discusses a partition for hypographs of continuous functions whose elements form a tree. This partition is the key to the discussion of the thick thin partition of the next section. Section 6 presents the thick thin partition of the hypograph relevant to the proof. It then shows that the number of components of the thin part is linearly bounded by the energy. Moreover, each component lies in a suitable annulus. In Section 7 we define a notion of tame geodesics in holomorphic annuli and discuss their properties. Roughly speaking, tame geodesics are those that do not wrap around the annulus too quickly. In Section 8 we prove Theorem 4.2. In Section 9 we deduce all the theorems of the introduction from Theorem 4.2. Finally, in Section 10 we prove Proposition 1.3.

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2. PRELIMINARIES ON CONFORMAL GEOMETRY.

Let (I, j) be a compact doubly connected surface with complex structure j . The **modulus** of (I, j) , denoted by $Mod(I, j)$ or $Mod(I)$ when the complex structure is clear from the context, is the unique real number $r > 0$ such that (I, j) is conformally equivalent to $[0, r] \times S^1$ (see [6]). Here S^1 is taken to be a standard circle of length 2π . In the sequel, we denote by h_{st} the unique flat metric on I with respect to which it has circumference 2π .

Let (I, h) be doubly connected with Riemannian metric h . We call global cylindrical coordinates (ρ, θ) on I ,

$$a \leq \rho \leq b, \quad 0 \leq \theta < 2\pi,$$

axially symmetric if

$$(9) \quad h = d\rho^2 + h_\theta(\rho)^2 d\theta^2.$$

We say h is **axially symmetric** if I has axially symmetric coordinates. In this case, the conformal length of (I, h) is given by

$$\text{Mod}(I, h) = \int_a^b \frac{1}{h_\theta(\rho)} d\rho.$$

Definition 2.1. Let I be a doubly connected surface of conformal length L . Then there is a holomorphic map $f : [0, L] \times S^1 \rightarrow I$ unique up to a rotation and a holomorphic reflection. For real numbers $a \leq b \in [0, L]$ with $a \leq L - b$, we write

$$S(a, b; I) := f([a, b] \times S^1) \subset I,$$

and

$$C(a, b; I) := S(a, L - b; I).$$

Note that composing f with a holomorphic reflection of $[0, L] \times S^1$ replaces $S(a, b)$ with $S(L - b, L - a)$. The expression $C(a, a)$ is independent of the choice of f . A **subcylinder** of I is a set of the form $I' = S(a, b; I)$.

Definition 2.2. Let U be a Riemann surface biholomorphic to the unit disk D_1 . Let h be a conformal metric on U and let $z \in U$. Then there is a biholomorphism $\phi : U \rightarrow D_1$ with $\phi(z) = 0$, unique up to rotation. The **conformal radius of U viewed from z** is defined to be

$$r_{\text{conf}}(U, z; h) := 1/\|d\phi(z)\|_h.$$

Note that $r_{\text{conf}}(U, z; h)$ is not conformally invariant, since it depends on the metric at z . However, let ν_h denote the volume form of h , let μ be an absolutely continuous measure on U and denote by $\frac{d\mu(z)}{d\nu_h}$ the Radon-Nikodym derivative. Then the expression $\frac{d\mu(z)}{d\nu_h} r_{\text{conf}}^2(z)$ is conformally invariant.

The cases of interest for us will be conformal radii of geodesic disks with metrics of constant curvature K , viewed from their center. In these cases, the metric can be written in polar coordinates as

$$(10) \quad h = d\rho^2 + h_\theta^2(\rho) d\theta^2,$$

where

$$(11) \quad h_\theta(\rho) = \begin{cases} \sinh(\rho), & K = -1, \\ \rho, & K = 0, \\ \sin(\rho), & K = 1. \end{cases}$$

So, the conformal radius of $B_r(p)$ viewed from p is given by

$$r_{conf} = \exp(f(r))$$

where f is the function defined by

$$f'(r) = \frac{1}{h_\theta(r)}, \quad f(r) = \log(r) + O(r) \text{ as } r \rightarrow 0.$$

More explicitly,

$$(12) \quad f(r) = \log(r) + \int_0^r \left(\frac{1}{h_\theta(\rho)} - \frac{1}{\rho} \right) d\rho.$$

It follows from equation (12) that

$$(13) \quad r_{conf} \geq r, \quad K = 0, 1,$$

and for any κ there exists a constant $c > 0$ such that

$$(14) \quad r_{conf} \geq cr, \quad K = -1, \quad r < \kappa.$$

Definition 2.3. For any Riemann surface $\Sigma = (\Sigma, j)$, write $\overline{\Sigma} := (\Sigma, -j)$. The **complex double** is the Riemann surface

$$\Sigma_{\mathbb{C}} := \Sigma \cup \overline{\Sigma},$$

where the surfaces are glued together along the boundary by the identity. The complex structure on $\Sigma_{\mathbb{C}}$ is the unique one which coincides with j and with $-j$ when restricted suitably. $\Sigma_{\mathbb{C}}$ is endowed with a natural antiholomorphic involution and for any $z \in \Sigma_{\mathbb{C}}$ we denote by \bar{z} the image of z under this involution. For more details about these constructions see [3].

Remark 2.4. Note that for any connected Riemann surface Σ , $\Sigma_{\mathbb{C}}$ is connected if and only if $\partial\Sigma \neq \emptyset$. Also, for any Σ , $\partial\Sigma_{\mathbb{C}} = \emptyset$.

Definition 2.5. Let $I \subset \Sigma_{\mathbb{C}}$ be doubly connected and conjugation invariant. If

$$S\left(0, \frac{1}{2}ModI; I\right) = \overline{S\left(\frac{1}{2}ModI, ModI; I\right)},$$

we say the conjugation on I is **latitudinal**. If for each $a, b \in [0, ModI]$,

$$S(a, b; I) = \overline{S(a, b; I)},$$

we say the conjugation on I is **longitudinal**.

Lemma 2.6. *Let $I \subset \Sigma_{\mathbb{C}}$ be doubly connected and conjugation invariant. Then the conjugation on I is either latitudinal or longitudinal.*

Proof. The lemma is a consequence of the classification of the holomorphic automorphisms of the annulus and the fact that the composition of two anti-holomorphic automorphisms is holomorphic. \square

For later reference we conclude with a statement of the thick thin decomposition for surfaces of genus $g > 1$. In the following we assume the surfaces are endowed with their unique metric h of constant curvature -1 .

Theorem 2.7. [5, 4.1.1] *Let S be a compact Riemann surface of genus $g \geq 2$, and let $\gamma_1, \dots, \gamma_m$ be pairwise disjoint simple closed geodesics on S . Then the following hold:*

- (a) $m \leq 3g - 3$.
- (b) *There exist simple closed geodesics $\gamma_{m+1}, \dots, \gamma_{3g-3}$, which, together with $\gamma_1, \dots, \gamma_m$, decompose S into pairs of pants.*
- (c) *The collars*

$$\mathcal{C}(\gamma_i) = \{p \in S \mid \text{dist}(p, \gamma_i) \leq w(\gamma_i)\}$$

of widths

$$w(\gamma_i) = \sinh^{-1} \left(1 / \sinh \left(\frac{1}{2} \ell(\gamma_i) \right) \right)$$

are pairwise disjoint for $i = 1, \dots, 3g - 3$.

- (d) *Each $\mathcal{C}(\gamma_i)$ is isometric to the cylinder $[-w(\gamma_i), w(\gamma_i)] \times S^1$ with the Riemannian metric*

$$d\rho^2 + \frac{\ell^2(\gamma_i) \cosh^2(\rho)}{4\pi^2} d\theta^2.$$

Denote by $\text{InjRad}(S; h, p)$ the radius of injectivity of S at $p \in S$, i.e. the supremum of all r such that $B_r(p)$ is an embedded disk. If h or S is clear from the context, we may omit it from the notation.

Theorem 2.8. [5, 4.1.6] *Let β_1, \dots, β_k be the set of all simple closed geodesics of length $\leq \sinh^{-1} 1$ on S . Then $k \leq 3g - 3$ and the following hold.*

- (a) *The geodesics β_1, \dots, β_k are pairwise disjoint.*
- (b) *$\text{InjRad}(S; p) > \sinh^{-1} 1$ for all $p \in S - (\mathcal{C}(\beta_1) \cup \dots \cup \mathcal{C}(\beta_k))$.*
- (c) *If $p \in \mathcal{C}(\beta_i)$, and $d = \text{dist}(p, \partial \mathcal{C}(\beta_i))$, then*

$$(15) \quad \sinh(\text{InjRad}(S; p)) = \cosh \frac{1}{2} \ell(\beta_i) \cosh d - \sinh d.$$

3. THICK THIN MEASURE

For the rest of the discussion, fix constants $c_1, c_2, c_3, \delta_1, \delta_2 > 0$. Without loss of generality we will assume that $c_3 \leq 1$ and that $\delta_2 < \delta_1$. Given a metric h on a measured Riemann surface (Σ, j, μ) we denote by $\frac{d\mu}{d\nu_h}$ the Radon Nikodym derivative of μ with respect to ν_h , the volume form induced by h .

Definition 3.1. Let Σ be a Riemann surface. A subset $S \subset \Sigma_{\mathbb{C}}$ is said to be **clean** if either $S = \overline{S}$ or $S \cap \overline{S} = \emptyset$.

Definition 3.2. Let (Σ, j) be a Riemann surface, possibly bordered. Let μ be a finite measure on Σ and extend μ to a measure on $\Sigma_{\mathbb{C}}$ by reflection i.e.

$$\mu(U) := \mu(\overline{U}),$$

for $U \subset \overline{\Sigma}$ a measurable set. μ will be called **thick thin** if it satisfies the following conditions:

- (a) μ is absolutely continuous and has a continuous density $\frac{d\mu}{d\nu_h}$, where h is any Riemannian metric on $\Sigma_{\mathbb{C}}$.
- (b) **Gradient inequality.** Let $U \subset \Sigma_{\mathbb{C}}$ be a simply connected domain and let $z \in U$. Then for any conformal metric h on $(\Sigma_{\mathbb{C}}, j)$,

$$\mu(U) < \delta_1 \quad \Rightarrow \quad \frac{d\mu}{d\nu_h}(z) \leq c_1 \frac{\mu(U)}{r_{conf}^2},$$

where $r_{conf} = r_{conf}(U, z; h)$.

- (c) **Cylinder inequality.** Let $I \subset \Sigma_{\mathbb{C}}$ be a doubly connected domain so that $Mod(I) > 2c_2$ and satisfying either $I \subset \Sigma$ or $I = \overline{I}$. Then

$$\begin{aligned} \mu\{I\} < \delta_2 &\Rightarrow \quad \mu\{C(t, t; I)\} \leq e^{-c_3 t} \mu\{I\}, \\ &\quad \forall t \in (c_2, \frac{1}{2}Mod(I)). \end{aligned}$$

Remark 3.3. Let μ be a thick thin measure on Σ , and let h be a conformal metric of constant curvature $K = 0, \pm 1$ on $\Sigma_{\mathbb{C}}$. By inequalities (13) and (14), there is a constant c'_1 depending linearly on c_1 such that for any $z \in \Sigma$ and $r \in (0, \min(\sinh^{-1}(1), InjRad(\Sigma; h, z)))$,

$$(16) \quad \mu(B_r(z; h)) < \delta_1 \Rightarrow \frac{d\mu}{d\nu_h}(z) \leq c'_1 \frac{\mu(B_r(z; h))}{r^2}.$$

Remark 3.4. Let Σ, μ and h be as in Remark 3.3. We will apply the gradient inequality in the following way. For any point $z \in \Sigma_{\mathbb{C}}$, let

$d = \frac{d\mu}{d\nu_h}(z)$ and let

$$(17) \quad r_d := \sqrt{\frac{c'_1 \delta_1}{d}}.$$

Suppose

$$r_d \in (0, \min(\sinh^{-1}(1), \text{InjRad}(\Sigma; h, z))).$$

Then

$$\mu(B_{r_d}(z; h)) \geq \delta_1.$$

Moreover, we have

$$\frac{d\mu}{d\nu_h}(z) \leq c'_1 \frac{\mu(B_r(z; h))}{r^2}, \quad r \leq r_d.$$

To simplify our formulas, we always scale μ so that $c'_1 \delta_1 = 1$.

We denote by $\mathcal{M} = \mathcal{M}(c_1, c_2, c_3, \delta_1, \delta_2)$ the family of measured Riemann surfaces (Σ, j, μ) such that μ is thick-thin.

Lemma 3.5. *There is a constant a with the following significance. Let $(\Sigma, j, \mu) \in \mathcal{M}$. Let $I \subset \Sigma_{\mathbb{C}}$ be clean and doubly connected, and let $h = h_{st}$. Suppose $\mu(I) < \delta_2$. Let $z \in C(c_2 + \pi, c_2 + \pi; I)$ be a point with cylindrical coordinates*

$$(s, t) \in \left[-\frac{1}{2} \text{Mod}(I) + c_2 + \pi, \frac{1}{2} \text{Mod}(I) - c_2 - \pi \right] \times S^1.$$

Then,

$$(18) \quad \frac{d\mu}{d\nu_h}(z) < a e^{-c_3(\frac{1}{2} \text{Mod}(I) - |s|)} \mu(I).$$

Proof. Combining the gradient inequality and the cylinder inequality,

$$(19) \quad \begin{aligned} \frac{d\mu}{d\nu_h}(z) &\leq \frac{c_1}{\pi^2} \mu([s - \pi, s + \pi] \times S^1) \\ &\leq \frac{c_1}{\pi^2} \mu([-|s| - \pi, |s| + \pi] \times S^1) \\ &\leq \frac{c_1}{\pi^2} e^{-c_3(\frac{1}{2} \text{Mod}(I) - \pi - |s|)} \mu(I). \end{aligned}$$

□

4. A PRIORI BOUND

Definition 4.1. Let (M, h) be a Riemannian manifold, let N be a totally geodesic submanifold possibly with boundary, and let h_N be the induced metric on N . Let $p \in N$. Define the **segment width** by

$$\begin{aligned} SegWidth(N, p; h) &= \\ &= \sup\{s > 0 \mid B_r(p; (M, h)) \cap N = B_r(p; (N, h_N)) \text{ for all } r < s\}. \end{aligned}$$

In the following, an **embedded geodesic** is a one-dimensional totally geodesic submanifold possibly with boundary. Let (M, h) be a Riemannian manifold. For γ an embedded geodesic, we denote by $d\ell_h$ the line element, or volume form, of the induced metric on γ . Now, consider the special case when M is a Riemann surface Σ . Let μ be a measure and h a conformal metric on Σ . Define

$$d\ell_\mu = \sqrt{\frac{d\mu}{d\nu_h}} \Big|_\gamma d\ell_h.$$

It is easy to see that $d\ell_\mu$ is independent of h . If γ is compact, define

$$\ell_\mu(\gamma) = \int_\gamma d\ell_\mu.$$

Let Σ be a Riemann surface possibly with boundary. For the rest of the paper, denote by h_{can} the unique conformal metric on $\Sigma_{\mathbb{C}}$ satisfying the following conditions. If $g \neq 0$, then h_{can} has constant curvature ± 1 . If $g = 0$, then h_{can} has constant curvature 0 and $\nu_{h_{can}}(\Sigma) = 1$.

Theorem 4.2. *There are constants b_1 and b_2 with the following significance. Let $(\Sigma, \mu) \in \mathcal{M}$, let $\gamma \subset \Sigma_{\mathbb{C}}$ be a compact embedded conjugation invariant geodesic, and let $k \geq 1$ be a constant such that for any $x \in \gamma$,*

$$(20) \quad SegWidth(\gamma, x; h_{can}) > \frac{1}{k} InjRad(\Sigma_{\mathbb{C}}; h_{can}, x).$$

Then

$$(21) \quad \ell_\mu(\gamma) \leq k^2 \{b_1 \mu(\Sigma_{\mathbb{C}}) + b_2 genus(\Sigma_{\mathbb{C}})\}.$$

For the rest of this discussion up to and including the proof of Theorem 4.2, we fix γ and k .

Remark 4.3. Recall the definition of c'_1 from Remark 3.3. In the proof of Theorem 4.2, without loss of generality, we may assume the constants c'_1, δ_1 , pertaining to the definition of thick-thin satisfy $c'_1 \delta_1 = 1$. This is true for two reasons. First, for $\tilde{c}_1 \geq c_1$ and $\tilde{\delta}_1 \leq \delta_1$, we have

$$\mathcal{M}(g, c_1, c_2, c_3, \delta_1, \delta_2) \subset \mathcal{M}(g, \tilde{c}_1, c_2, c_3, \tilde{\delta}_1, \delta_2).$$

Such $\tilde{c}_1, \tilde{\delta}_1$, can always be chosen so that $\tilde{c}'_1 \tilde{\delta}_1 = 1$. However, this will not yield the optimal constant c for a given c_1, δ_1 . To obtain the optimal value of c , it is useful to note that for $\lambda > 0$, the map

$$\mathcal{M}(c_1, c_2, c_3, \delta_1, \delta_2) \rightarrow \mathcal{M}(c_1, c_2, c_3, \lambda\delta_1, \lambda\delta_2)$$

given by $(\Sigma, j, \mu) \mapsto (\Sigma, j, \lambda\mu)$ scales the constants b_i for $i = 1, 2$, by $b_1 \mapsto b_1/\sqrt{\lambda}$ and $b_2 \mapsto \sqrt{\lambda}b_2$.

For any metric n on γ , denote by $d\ell_n$ the line element. By definition,

$$\ell_\mu(\gamma) = \int_{x \in \gamma} \frac{d\ell_\mu}{d\ell_n} d\ell_n(x).$$

We derive Theorem 4.2 by studying the graph of the function

$$g := \ln \frac{d\ell_\mu}{d\ell_n} : \gamma \rightarrow (-\infty, \infty)$$

for a convenient choice of metric n . We define n as follows. For any $x \in \gamma$, let

$$r(x) := \min(\sinh^{-1}(1), \text{InjRad}(\Sigma; h_{can}, x)).$$

It turns out that for dealing with higher genus, where there is no a priori bound on the radius of injectivity of Σ , it is convenient to use the metric $n = \frac{1}{r(x)} h_{can}|_\gamma$.

We use the normalized metric n only on γ . On $\Sigma_{\mathbb{C}}$ we continue to use the standard metric h_{can} . To translate from estimates in terms of the one to estimates in terms of the other metric, we will use the following lemma.

Lemma 4.4. *Let $x_1, x_2 \in \gamma$ such that $d_\gamma(x_1, x_2; h_{can}) \leq r(x_1)/2$. Then*

$$(22) \quad \frac{2d_\gamma(x_1, x_2; h_{can})}{3r(x_1)} \leq d_\gamma(x_1, x_2; h_n) \leq \frac{2d_\gamma(x_1, x_2; h_{can})}{r(x_1)}.$$

Lemma 4.5. *For all t such that $r(\gamma(t))$ is differentiable,*

$$(23) \quad \frac{dr(\gamma(t))}{dt} \leq 1.$$

Proof. If h_{can} has curvature -1 , inequality (4.5) follows from Theorem 2.8(b) and (c). If h_{can} has non-negative curvature, then $\frac{dr(\gamma(t))}{dt} = 0$. \square

Proof of Lemma 4.4. Write $\Delta x = d_\gamma(x_1, x_2; h_{can})$. Parameterize γ by h_{can} -length so that $\gamma(0) = x_1$ and $\gamma(\Delta x) = x_2$. It is easy to see that

$r(\gamma(t))$ is piecewise smooth and thus differentiable almost everywhere with respect to t . Applying Lemma 4.5 we calculate

$$\begin{aligned}
(24) \quad d_\gamma(x_1, x_2; h_n) &= \left| \int_0^{\Delta x} \frac{1}{r(\gamma(t))} dt \right| \\
&\leq \frac{\Delta x}{\inf_{t \in [0, \Delta x]} r(\gamma(t))} \\
&\leq \frac{\Delta x}{r(x_1) - \Delta x \operatorname{ess\,sup}_{t \in [0, \Delta x]} \left| \frac{dr(\gamma(t))}{dt} \right|} \\
&\leq \frac{2\Delta x}{r(x_1)}.
\end{aligned}$$

For the last inequality we have used (4.5). The upper bound of estimate (22) follows. A similar argument gives the lower bound. \square

Definition 4.6. Denote

$$D := \{(x, t) \in \gamma \times [\ln 2k, \infty) \mid \ln 2k \leq t \leq g(x)\}.$$

For any $(x, t) \in D$, denote $B(x, t) := B_{e^{-t}r(x)}(x; \Sigma, h_{can})$.

Lemma 4.7. For $(x, t) \in D$, we have $\mu(B(x, t)) \geq \delta_1$.

Proof. Since $k \geq 1$, we have $t > 0$. Therefore, $B(x, t)$ is an embedded disk. By Remark 3.4, $\mu(B(x, t)) \geq \delta_1$. \square

Lemma 4.8. Let $(x_i, t_i) \in D$ for $i = 1, 2$. Suppose

$$d_\gamma(x_1, x_2; h_n) > 2(e^{-t_1} + e^{-t_2}).$$

Then $B(x_1, t_1) \cap B(x_2, t_2) = \emptyset$.

Proof. By Lemma 4.4 we have

$$d_\gamma(x_1, x_2; h_{can}) \geq e^{-t_1}r(x_1) + e^{-t_2}r(x_2).$$

Since $t_i \geq \ln 2k$, the assumption of Theorem 4.2 implies that

$$\operatorname{SegWidth}(\gamma, x_i; h_{can}) \geq 2e^{-t_i}r(x_i).$$

The claim now follows. \square

5. PARTITIONS OF HYPOGRAPHS

Let γ be a 1-dimensional manifold, let $f : \gamma \rightarrow [0, \infty)$ be a continuous function, and let E be the hypograph of f . That is, E is the set of points under the graph of f in $\gamma \times [0, \infty)$. In this section we introduce a binary relation on subsets of E , which should be thought of intuitively as the relation of lying above. We prove two basic theorems about this order relation. Theorem 5.7 states that for any partition P of E

into connected subsets by intersecting E with horizontal segments, the binary relation on the elements of P is a tree-like partial order. See Figure 4. Theorem 5.21 states that there is a particular such partition, denoted \mathcal{T}_E , such that the branchings in the tree associated with \mathcal{T}_E correspond to local minima in the graph of f . See Figure 5. After proving these theorems, we show that continuity of f allows us to control the number of elements of \mathcal{T}_E by the number of its maximal elements. Note that in general \mathcal{T}_E might be infinite, and if f is not continuous, there might not be any maximal elements.

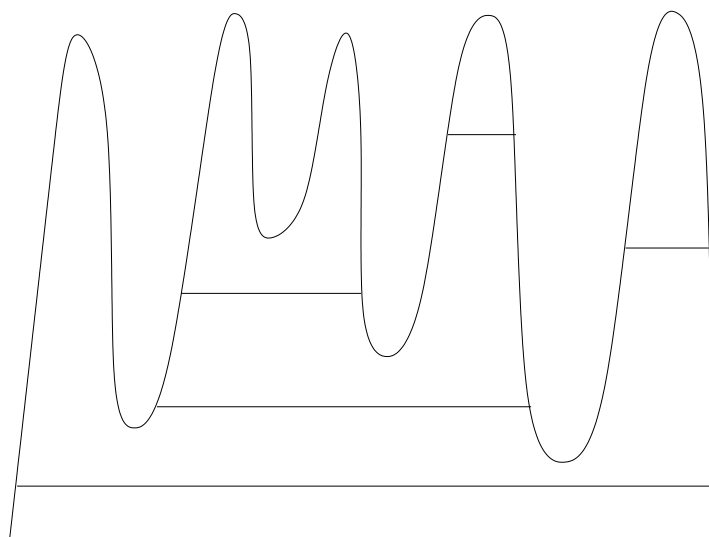


FIGURE 4.

5.1. A binary relation. Let γ be a compact 1-dimensional manifold with or without boundary. Write $X = \gamma \times [\xi, \infty)$ and denote by $p_1 : X \rightarrow \gamma$ and $p_2 : X \rightarrow [\xi, \infty)$ the canonical projections. Denote $X_t := \gamma \times \{t\}$. For any subset $S \subset X$ denote $S_t := X_t \cap S$. If $p_2(S) \subset [\xi, \infty)$ is bounded, denote

$$T_f(S) := \sup\{p_2(S)\},$$

$$T_i(S) := \inf\{p_2(S)\},$$

and $T(S) := T_f(S) - T_i(S)$.

Let $f : \gamma \rightarrow [\xi, \infty)$ be a continuous function. Denote the region under the graph of f by

$$E := \{y \in X | p_2(y) \leq f(p_1(y))\}.$$

For a topological space Y , denote by $\pi_0(Y)$ the set of path-connected components. An **E -segment** is an element of $\cup_{t \in [\xi, \infty)} \pi_0(E_t)$. For any

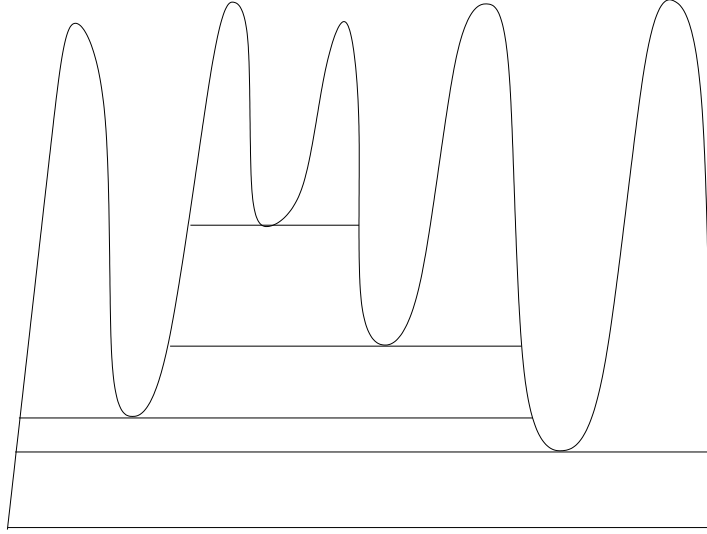


FIGURE 5.

$t \geq \xi$, and for any $x \in p_1(E_t)$ we denote by $e(x, t)$ the E -segment containing (x, t) .

Remark 5.1. It follows from the continuity of f that E is a closed set. So, all E -segments are closed. It also follows from the continuity of f that if e is an E -segment, x is not a boundary point of γ , and (x, t) is a boundary point of e , then $f(x) = t$.

We define a relation on the power set $P(E)$ as follows. Let $S_1, S_2 \subset E$. We say that $S_1 \leq_1 S_2$ if $p_1(S_2) \subseteq p_1(S_1)$. We say that $S_1 \leq_2 S_2$ if for any $t \in p_2(S_2)$ there is a $t' \in p_2(S_1)$ such that $t' \leq t$. Finally we say that $S_1 \leq S_2$ is $S_1 \leq_1 S_2$ and $S_1 \leq_2 S_2$.

The following properties of \leq are obvious and are stated without proof.

Lemma 5.2. (a) *The relation \leq is reflexive and transitive.*

(b) *Let $\Pi \subset P(E)$ be the collection of subsets of the form*

$$s_1 \times \{t\},$$

where $s_1 \subset \gamma$, $t \in [\xi, \infty)$. The restriction of \leq to Π is antisymmetric. Π contains the singletons of E and the E -segments.

(c) *For any two sets $S_1, S_2 \in P(E)$, $S_1 \leq S_2$ if and only if for any $(x, t) \in S_2$, $S_1 \leq \{(x, t)\}$*

Lemma 5.3. *Let $\xi \leq t \leq t_1, t_2$, let $x \in p_1(E_t)$, $x_i \in p_1(E_{t_i})$ for $i = 1, 2$ and denote $e_i = e(x_i, t_i)$.*

(a) *For any $t' \in [\xi, t]$, $e(x, t')$ is well defined.*

- (b) If $e(x_1, t) = e(x_2, t)$ then for any $t' \in [\xi, t]$, $e(x_1, t') = e(x_2, t')$.
- (c) For any $t' \in [\xi, t]$, $e(x, t') \leq e(x, t)$.
- (d) $e_1 \leq e_2$ if and only if $t_1 \leq t_2$ and $e(x_2, t_1) = e_1$.
- (e) If e_1 and e_2 are incomparable with respect to \leq , then

$$d(p_1(e_1), p_1(e_2)) > 0.$$

- (f) If $e(x_1, t_1) \leq e(x_2, t_2)$, then for any $t \in [t_1, t_2]$, $e(x_1, t_1) \leq e(x_2, t)$.

Proof. (a) We have $t' \leq t \leq f(x)$, so $(x, t') \in E$.

- (b) Let $S = p_1(e(x_1, t))$ then S is a segment which by assumption contains x_1 and x_2 . Since $S \times \{t\} \subset E_t$, for all $t' \in [\xi, t]$, and all $x \in S$, $t' \leq f(x)$. Therefore, $S \times \{t'\} \subset E_{t'}$. $S \times \{t'\}$ is connected and contains (x_i, t') for $i = 1, 2$. In particular $e(x_1, t') = e(x_2, t')$.
- (c) It is clear that $e(x, t') \leq_2 e(x, t)$. If $x' \in p_1(e(x, t))$, then $e(x', t) = e(x, t)$. By (b), $e(x', t') = e(x, t')$. In particular, $x' \in p_1(e(x, t'))$. Thus $e(x, t') \leq_1 e(x, t)$.
- (d) Assume first that $e_1 \leq e_2$. Then $t_1 \leq t_2$ by definition. Further, $x_2 \in p_1(e_1)$, so $e(x_2, t_1) \subset e_1$. But e_1 is an E -segment, so $e(x_1, t_2) = e_1$ as required. Assume now that $e(x_2, t_1) = e_1$, and $t_1 \leq t_2$. Then by (c), $e_1 \leq e(x_2, t_2)$.
- (e) Assume without loss of generality that $t_1 \leq t_2$ and denote $e'_2 = e(x_2, t_1)$. By Remark 5.1, E_{t_1} is closed. So, since e'_2 and e_1 are both connected components of E_{t_1} , we have that either $e'_2 = e_1$ or $d(p_1(e'_2), p_1(e_1)) > 0$. Thus, by (d), $d(p_1(e'_2), p_1(e_1)) > 0$. By (c), $e'_2 \leq e_2$. In particular, $p_1(e_2) \subset p_1(e'_2)$, so

$$d(p_1(e_1), p_1(e_2)) > d(p_1(e'_2), p_1(e_1)) > 0.$$

- (f) Using (d) twice, $e(x_1, t_1) \leq e(x_2, t_2)$ implies $e(x_2, t_1) = e(x_1, t_1)$, which implies $e(x_1, t_1) \leq e(x_2, t)$.

□

5.2. Tree-like partial order.

Definition 5.4. Let $S \subset E$. S is said to be **E -saturated** if S is a union of E -segments.

Remark 5.5. Clearly, any union or intersection of E -saturated sets is E -saturated. Moreover, a connected component of an E -saturated set and the complement in E of an E -saturated set are E -saturated.

Definition 5.6. Let S be a set. A tree-like order relation on S is a partial order relation \leq which satisfies for any $v, v_1, v_2 \in S$,

$$v_1 \leq v \text{ and } v_2 \leq v \quad \Rightarrow \quad v_1 \leq v_2 \text{ or } v_2 \leq v_1.$$

Theorem 5.7. *Let P be a collection of pairwise disjoint E -saturated connected sets. Then the restriction of \leq to P is a tree-like order relation.*

For the proof of Theorem 5.7, we first prove a few lemmas.

Lemma 5.8. *Let $S \subset E$ be connected. For any compact set $K \subset S$ there exists a point $(x, t) \in S$ such that $e(x, t) \leq K$.*

Proof. Using the compactness of K , choose $(x_1, t_1) \in K$ such that $t_1 = T_i(K)$. Let $L \subset p_1(S)$ be a connected compact subset containing $p_1(K)$. Using the continuity of f , choose $x_2 \in L$ such that

$$f(x_2) = \inf_{y \in L} f(y).$$

Since $x_2 \in p_1(S)$, and $S \subset E$, there exists $t_2 \leq f(x_2)$ such that $(x_2, t_2) \in S$. Choose i such that $t_i = \min(t_1, t_2)$ and set $(x, t) = (x_i, t_i)$. Clearly, $(x, t) \in S$. Since $t \leq t_2$, we have

$$L \times \{t\} \subset E.$$

So, since $L \times \{t\}$ is connected and contains (x, t) , we have

$$L \times \{t\} \subset e(x, t).$$

Therefore, since $p_1(K) \subset L$ and $t \leq t_1$, we have $e(x, t) \leq K$. □

Lemma 5.9. *Let $S_1 \subset E$ be E -saturated and let $S_2 \subset E$ be connected such that $S_1 \cap S_2 = \emptyset$. Suppose there exist $x \in \gamma$ and $t_1 < t_2 \in [\xi, \infty)$ such that $(x, t_i) \in S_i$ for $i = 1, 2$. Then $S_1 \leq S_2$.*

Proof. Let $e = e(x, t_1)$. Since S_1 is E -saturated, we have $e \subset S_1$. So,

$$(25) \quad e \cap S_2 = \emptyset.$$

Let $x_1, x_2 \in \gamma$ be the boundary points of $p_1(e)$. By Remark 5.1,

$$(26) \quad f(x_i) = t_1, \quad \text{for } i \in \{1, 2\} \text{ such that } x_i \notin \partial\gamma.$$

Define

$$r_i = \begin{cases} \{x_i\} \times (t_1, \infty), & x_i \notin \partial\gamma \\ \emptyset & x_i \in \partial\gamma. \end{cases}$$

By equation (26), the rays r_i are disjoint from E . In particular,

$$(27) \quad S_2 \cap r_i = \emptyset, \quad i = 1, 2.$$

Define disjoint open sets U_1 and U_2 by

$$U_2 = \{(x, s) \in X \mid x \in (x_1, x_2) \cup (\partial\gamma \cap \{x_1, x_2\}) \text{ and } s > t_1\},$$

$$U_1 = X \setminus \overline{U_2}.$$

Clearly,

$$U_1 \cup U_2 = X \setminus (e \cup r_1 \cup r_2).$$

So, by equations (25) and (27), $S_2 \subset U_1 \cup U_2$. Since $(x, t_2) \in S_2 \subset E$, we have $f(x) \geq t_2 > t_1$. Thus by equation (26), we have $x \notin \{x_1, x_2\} \setminus \partial\gamma$. So, by definition of U_2 , we have $(x, t_2) \in U_2$. Therefore, $S_2 \cap U_2 \neq \emptyset$. Since S_2 is connected, it follows that $S_2 \subset U_2$. So, $U_2 \leq S_2$. By definition of U_2 , we have $e \leq U_2$. Since $e \subset S_1$, we have $S_1 \leq e$. Combining the foregoing inequalities, we have

$$S_1 \leq e \leq U_2 \leq S_2,$$

which proves the lemma. \square

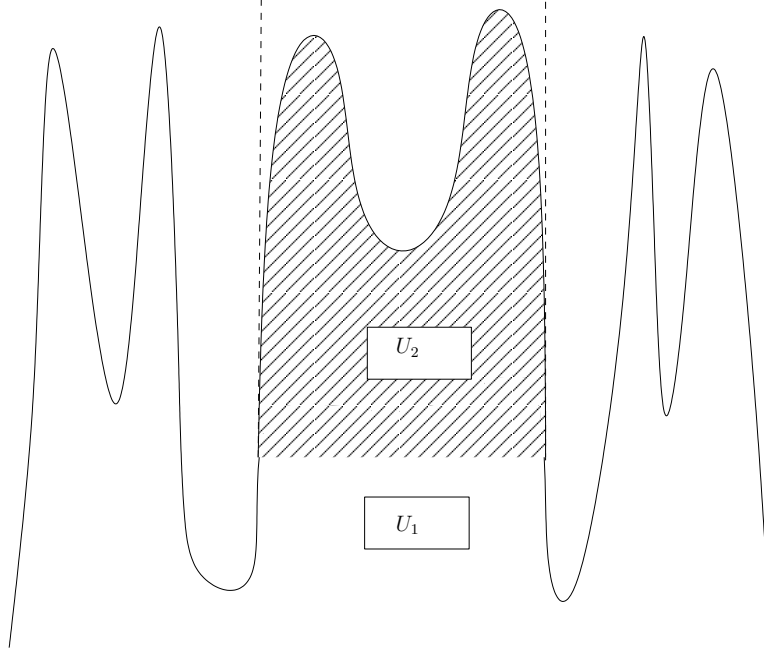


FIGURE 6.

Lemma 5.10. *Let $S \subset E$ be E -saturated and connected.*

- (a) *Let $e_1, e_2 \subset S$ be E -segments. If e_3 is an E -segment such that $e_1 \leq e_3 \leq e_2$, then $e_3 \subset S$.*
- (b) *For any $t_1 \leq t_2 \in p_2(S)$, $S_{t_1} \leq S_{t_2}$.*

Proof. (a) Suppose that $e_3 \not\subset S$. Since S is E -saturated it follows that

$$(28) \quad e_3 \cap S = \emptyset.$$

Take $S_1 = e_3$ and $S_2 = S$. Set $t_0 = p_2(e_1)$, $t_1 = p_2(e_3)$ and $t_2 = p_2(e_2)$. Since $e_1, e_2 \subset S$, by equation (28) we have $t_0 < t_1 < t_2$. Choose $x \in p_2(e_2)$. Then S_1, S_2, x, t_1, t_2 satisfy the hypotheses of Lemma 5.9. We conclude that $e_3 \leq S$. So,

$$p_2(e_1) = t_0 < t_1 = p_2(e_3) \leq T_i(S)$$

contradicting the assumption that $e_1 \subset S$.

- (b) By Lemma 5.8 with $K = S_{t_1} \cup S_{t_2}$, there exists $(x, t) \in S$ such that $e(x, t) \leq S_{t_i}$ for $i = 1, 2$. Since S is E -saturated, $e(x, t) \subset S$. Let $y \in p_1(S_{t_2})$. Since S is E -saturated, $e(y, t_2) \subset S$. In particular, $e(x, t) \leq e(y, t_2)$. So, by Lemma 5.3(f), $e(x, t) \leq e(y, t_1)$. By Lemma 5.3(c), $e(y, t_1) \leq e(y, t_2)$. Therefore, by ((a)) we have $e(y, t_1) \subset S$. Since $y \in p_1(S_{t_2})$ was arbitrary, it follows that $S_{t_1} \leq S_{t_2}$. \square

Corollary 5.11. *Let F be a collection of pairwise disjoint connected E -saturated sets. Then the restriction of the relation \leq to F is anti-symmetric.*

Proof. Let $S_1, S_2 \in F$ and let $x \in p_1(S_1)$. Assume $S_1 \leq S_2$ and $S_2 \leq S_1$. Then in particular, $p_1(S_1) = p_1(S_2)$ so that $x \in p_1(S_2)$. Therefore, there are $t_1, t_2 \in p_2(S_1)$ such that $(x, t_1) \in S_1$ and $(x, t_2) \in S_2$. Assume without loss of generality that $t_1 \leq t_2$. Since $S_2 \leq_2 S_1$ there is a $t \in p_2(S_2)$ with $t \leq t_1$. Since S_2 is connected, $p_2(S_2)$ is connected and so $[t, t_2] \subset p_2(S_2)$. In particular $t_1 \in p_2(S_2)$. By Lemma 5.10(b), $S_{2,t_1} \leq S_{2,t_2}$, so $(x, t_1) \in S_2$. Thus $S_1 \cap S_2$ is nonempty, and by the assumption on F , $S_1 = S_2$. \square

Lemma 5.12. *Let $S_1, S_2 \subset E$. Suppose that S_1 is E -saturated and connected, that S_2 is connected and that $S_1 \leq_2 S_2$. Then either $S_1 \leq S_2$ or*

$$p_1(S_1) \cap p_1(S_2) = \emptyset.$$

Proof. Suppose that there is a point $x_1 \in p_1(S_1) \cap p_1(S_2)$. Then there is a $t_1 \in p_2(S_2)$ such that $(x_1, t_1) \in S_2$. Let $(x_2, t_2) \in S_2$. By Lemma 5.8 with $K = (x_1, t_1) \cup (x_2, t_2)$, there is an $(x_3, t_3) \in S_2$ such that $e(x_3, t_3) \leq (x_i, t_i)$ for $i = 1, 2$. In particular,

$$p_1(e(x_3, t_3)) \cap p_1(S_1) \neq \emptyset.$$

Let $t \in p_2(S_1)$ such that $t \leq \min(t_1, t_3)$. Such a t exists by the assumption $S_1 \leq_2 S_2$. By Lemma 5.3(e) and the fact that $t \leq t_3$, $e(x_1, t) \leq e(x_3, t_3)$. By Lemma 5.10(b) and the fact that $t \leq t_1$, we

have $S_t \leq S_{t_1}$. In particular, $(x_1, t) \in S_1$. Since S_1 is E -saturated, $e(x_1, t) \subset S_1$. Therefore,

$$S_1 \leq e(x_1, t) \leq e(x_3, t_3) \leq \{(x_2, t_2)\}.$$

But (x_2, t_2) was an arbitrary point of S_2 , so the claim follows. \square

Lemma 5.13. *Let $S_1, S_2 \subset E$ be connected and let S_1 be E -saturated. Suppose $S_1 \leq_2 S_2$. If there is a nonempty set $S \subset E$ such that $S_i \leq S$ for $i = 1, 2$, then $S_1 \leq S_2$.*

Proof. By assumption, $p_1(S_1) \cap p_1(S_2) \neq \emptyset$. Therefore, by Lemma 5.12 $S_1 \leq S_2$. \square

Proof of Theorem 5.7. By Cor. 5.11, \leq is an order relation when restricted to P . By Lemma 5.13, this order is tree-like. \square

5.3. Equivalence relation.

Definition 5.14. A **branching point** is a point $(x, t) \in \partial E$ that is contained in an open segment $s \subset e(x, t)$ such that $\partial s \subset E^\circ$.

Definition 5.15. Let $(x_i, t_i) \in E$ for $i = 1, 2$. If $t_1 \leq t_2$, we say that $(x_1, t_1) \sim (x_2, t_2)$ if the following two conditions hold:

- (a) $e(x_1, t_1) \leq e(x_2, t_2)$.
- (b) The rectangle $R = p_1(e(x_1, t_1)) \times [t_1, t_2]$ contains no branching points.

If $t_2 < t_1$, we reverse the roles of t_1 and t_2 .

Lemma 5.16. *\sim is an equivalence relation.*

Before proving Lemma 5.16, we prove the following preparatory lemma.

Lemma 5.17. *Let $t_1 > \xi$ and $t_2 \geq t_1$. Let $x \in p_1(E_{t_2})$. Let $R = p_1(e(x, t_1)) \times [t_1, t_2]$ and $\overline{R} = p_1(e(x, t_1)) \times [t_1, t_2]$. Assume that R contains no branching points.*

- (a) $(\overline{R} \cap E)_{t_2}$ is connected.
- (b) The order \leq on the set of E -segments contained in \overline{R} is linear.
- (c)

$$(p_1(e(x, t_2)) \times [t_2, \infty)) \cap E = (p_1(e(x, t_1)) \times [t_2, \infty)) \cap E$$

Proof. (a) We prove this by contradiction. Choose an orientation on γ so that intervals between points on γ are well defined. Let e_1 and e_2 be distinct connected components of $(\overline{R} \cap E)_{t_2}$. Let (x_1, t_2) and (x_2, t_2) be boundary points of e_1 and e_2 respectively

such that the segment $(x_1, x_2) \times \{t_2\}$ is contained in \overline{R} and is disjoint from both e_1 and e_2 . By continuity of f , choose $x_3 \in [x_1, x_2]$ where $f|_{[x_1, x_2]}$ obtains its minimum and let $t_3 = f(x_3)$. Since

$$x_3 \in [x_1, x_2] \subset p_1(\overline{R}) = p_1(e(x, t_1)),$$

it follows that $t_3 = f(x_3) \geq t_1$. Since

$$[x_1, x_2] \times t_2 \cap (X \setminus E)_{t_2} \neq \emptyset,$$

there is an $x' \in (x_1, x_2)$ such that $f(x') < t_2$, which implies that $t_3 < t_2$. For each $x' \in [x_1, x_2]$, $f(x') \geq t_3$. Therefore,

$$[x_1, x_2] \times \{t_3\} \subset e(x, t_3) \subset E.$$

On the other hand, since $f(x_i) = t_2 > t_3$ for $i = 1, 2$, we have that $x_3 \in (x_1, x_2)$. Furthermore, by continuity of f and the fact that $t_3 \geq t_1 > \xi$, we have $(x_1, t_3), (x_2, t_3) \in E^o$. Thus (x_3, t_3) is a branching point. Since $t_2 > t_3 \geq t_1$, (x_3, t_3) is contained in R contradicting the assumption.

- (b) Let $e_1 = e(x_1, t'_1)$ and $e_2 = e(x_2, t'_2)$ for $(x_1, t'_1), (x_2, t'_2) \in \overline{R}$. Suppose without loss of generality that $t'_1 \leq t'_2$, then by Lemma 5.3(d) it suffices to show that $e(x_1, t'_1) = e(x_2, t'_1)$. Since $e_i \in \overline{R}$, it follows that $e(x, t_1) \leq e(x_i, t'_1)$ for $i = 1, 2$. So, $e(x_i, t'_1)$ are connected components of $(R \cap E)_{t'_1}$. The claim now follows immediately from (a).
- (c) We show the less obvious inclusion. Let

$$(x', t') \in (p_1(e(x, t_1)) \times [t_2, \infty)) \cap E.$$

By (a),

$$e(x', t_2) = e(x, t_2).$$

In particular,

$$x' \in p_1(e(x, t_2)).$$

□

Remark 5.18. It is immediate from the definition of a branching point that the converse to 5.17(a) is also true. Namely, if there is a $t \in (t_1, t_2)$ for which R_t contains a branching point, then there is a $t' > t$ with $t' \in p_2(R \cap E)$ such that $R_{t'}$ has at least two components. It is clear that t' can be taken arbitrarily close to t .

Proof of Lemma 5.16. Suppose $(x_1, t_1) \sim (x_2, t_2)$ and $(x_2, t_2) \sim (x_3, t_3)$. We wish to prove that $(x_1, t_1) \sim (x_3, t_3)$. Without loss of generality we assume $t_1 \leq t_3$. For $i = 1, 2, 3$, denote $e_i = e(x_i, t_i)$, $c_i = p_1(e_i)$ and let $R = c_1 \times [t_1, t_3]$. We need to prove that $e_1 \leq e_3$ and that R contains no branching points.

We distinguish between the three possibilities for the order of t_1, t_2 and t_3 . We start with the case $t_2 \leq t_1 \leq t_3$. Then $R \subset c_2 \times [t_2, t_3)$ and thus R contains no branching points. By Lemma 5.17(b), e_1 and e_3 are comparable, and since $t_1 \leq t_3$, we have $e_1 \leq e_3$.

If $t_1 \leq t_2 \leq t_3$, then by Lemma 5.17(c),

$$R \cap E \subset p_1(e_1) \times [t_1, t_2) \cup p_1(e_2) \times [t_2, t_3),$$

and thus R contains no branching points. Furthermore,

$$e_1 \leq e_2 \leq e_3$$

by assumption.

If $t_1 \leq t_3 \leq t_2$, then $R \subset c_1 \times [t_1, t_2)$. So, there are no branching points in R . By Lemma 5.17(b), e_1 and e_3 are comparable. Since $t_1 \leq t_3$, it follows that $e_1 \leq e_3$. \square

5.4. Tree-like partition.

Definition 5.19. A subset $S \subset X$ is **closed from above** if for any $x \in \gamma$ the intersection $\{x\} \times [\xi, \infty) \cap S$ is closed from the right.

Remark 5.20. Any finite union, intersection and relatively closed subset of sets that are closed from above is closed from above.

Let \mathcal{T}_E denote the partition of E into \sim equivalence classes.

Theorem 5.21. *Each $c \in \mathcal{T}_E$ satisfies following properties.*

- (a) c is E -saturated and connected.
- (b) For any $c_1 \neq c \in \mathcal{T}_E$ such that $c \leq c_1$, there exists a $c_2 \in \mathcal{T}_E$ that is incomparable to c_1 and such that $c \leq c_2$.
- (c) Let $c' \subset c$ be E -saturated, connected and closed from above. Let $S \subset E$ be disjoint from c' . Then

$$c' \leq S \Rightarrow c'_{T_f(c')} \leq S.$$

In Figure 7 the shaded and white parts correspond to different elements in a partition of E . The right side of the figure shows what part (b) of Theorem 5.21 rules out. The left side shows what is ruled out by part (c).

Corollary 5.22. *The restriction of \leq to \mathcal{T}_E is a tree-like order relation.*

Proof. The corollary follows from Theorem 5.21(a) and Theorem 5.7. \square

Lemma 5.23. *Let $c \subset E$ be an equivalence class under \sim . Then c is closed from above.*

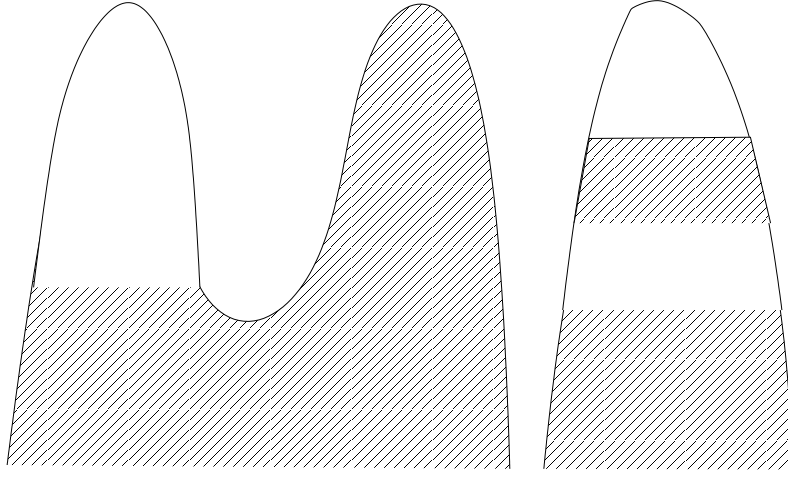


FIGURE 7.

Proof. Let $x \in p_1(c)$. Choose $t \geq \xi$ such that $(x, t) \in c$. Let

$$t_1 = \sup\{s \in [\xi, \infty) \mid (x, s) \in c\}.$$

Since E is closed and $c \subset E$, we have $(x, t_1) \in E$. It suffices to show that $(x, t_1) \in c$. Suppose it is not. By Lemma 5.3(c), $e(x, t) \leq e(x, t_1)$. So, the rectangle $R = p_1(e(x, t)) \times [t, t_1]$ must contain a branching point (x_1, t_2) . But then for any $t_3 \in (t_2, t_1)$, (x, t_3) is not contained in c . This contradicts the definition of t_1 . \square

Lemma 5.24. *Let $S \subset E$ be E -saturated and contained in a single \sim equivalence class. Then S_t is a single E -segment for all $t \in p_2(S)$.*

Proof. Since S is E -saturated, S_t is union of E -segments. Suppose $e(x_1, t), e(x_2, t) \subset S_t$. By assumption, $(x_1, t) \sim (x_2, t)$, so by definition of \sim , we have $e(x_1, t) \leq e(x_2, t)$ and $e(x_2, t) \leq e(x_1, t)$. Thus $e(x_1, t) = e(x_2, t)$ by Lemma 5.2(b). Therefore, S_t is a single E -segment. \square

Lemma 5.25. *Let $S \subset E$ be E -saturated, connected, closed from above and contained in a single \sim equivalence class. Then $T_f(S) \in p_2(S)$.*

Proof. By Lemma 5.24, S_t is a single E -segment for each $t \in p_2(S)$. So, by Remark 5.1 $p_1(S_t)$ is closed. Since γ is compact, so is $p_1(S_t)$. By Lemma 5.10(b), $p_1(S_{t'}) \subset p_1(S_t)$ for all $t' \geq t$. So $p_1(S_t)$ for $t \in p_2(S)$ is a nested family of compact non-empty sets. Therefore, we may choose

$$x \in \bigcap_{t \in p_2(S)} p_1(S_t).$$

So,

$$\{x\} \times p_2(S) = (\{x\} \times [\xi, \infty)) \cap S.$$

Therefore, since S is closed from above, $(x, T_f(S)) \in S$, and $T_f(S) \in p_2(S)$. \square

Lemma 5.26. *Let $S \subset E$ be E -saturated, connected, closed from above and contained in a single \sim equivalence class. Then for any $T \subset E \setminus S$,*

$$S \leq T \Rightarrow S_{T_f(S)} \leq T.$$

Proof. Let $(x, t) \in T$. We show first that $t > T_f(S)$. By assumption there is a $t_1 \in p_2(S)$ such that $(x, t_1) \in S$ and a $t_2 \in p_2(S)$ such that $t_2 \leq \min\{t, t_1\}$. By 5.10(b), $(x, t_2) \in S$. Assume now that $t \leq T_f(S)$. Then we have that

$$R = p_1(e(x, t_2)) \times [t_2, t) \subset p_1(S) \times (T_i(S), T_f(S)),$$

and thus R contains no branching point. By Lemma 5.25, there is an $x' \in \gamma$ such that $(x', T_f(S)) \in S$. By Lemma 5.10(b), $e(x', t) \subset S$. Thus by Lemma 5.24, $e(x', t) = e(x, t)$. In particular, $(x, t) \in S$ contradicting the assumption that $T \subset E \setminus S$. Thus $t > T_f(S)$, so $e(x, T_f(S))$ is well defined. By Lemma 5.24, $e(x, T_f(S)) = S_{T_f(S)}$, so by Lemma 5.3(d), $S_{T_f(S)} \leq e(x, t) \leq (x, t)$. Since $(x, t) \in T$ was arbitrary, the lemma follows. \square

Lemma 5.27. *Let $c \in \mathcal{T}_E$, and let $t_1, t_2 \in p_2(c)$ satisfy $t_1 \leq t_2$. Let $x_2 \in p_1(c_{t_2})$. Then $\{x_2\} \times [t_1, t_2] \subset c$.*

Proof. Let $x_1 \in p_1(c_{t_1})$. Then $(x_1, t_1) \sim (x_2, t_2)$. By definition of \sim , $e(x_1, t_1) \leq e(x_2, t_2)$. Let $t \in [t_1, t_2]$. Then by Lemmas 5.3(c) and 5.3(f),

$$e(x_1, t_1) \leq e(x_2, t) \leq e(x_2, t_2).$$

Let $R = p_1(e(x_2, t)) \times [t, t_2]$. Then $R \subset p_1(e(x_1, t_1)) \times [t_1, t_2]$. Therefore R contains no branching points. Thus $(x_2, t) \sim (x_2, t_2)$. The lemma follows. \square

Lemma 5.28. *Let $c \in \mathcal{T}_E$, and $(x_i, t_i) \in c$ for $i = 1, 2$. Suppose $t_1 \leq t_2$. There exists a path $\rho \subset c$ connecting (x_1, t_1) and (x_2, t_2) such that $p_2(\rho) \subset [t_1, t_2]$.*

Proof. Let $\rho' = \{x_2\} \times [t_1, t_2]$. By Lemma 5.27, $\rho' \subset c$. By definition of \sim , $e(x_1, t_1) \leq e(x_2, t_2)$. Thus ρ' connects the path connected sets $e(x_1, t_1)$ and $e(x_2, t_2)$. So, choose

$$\rho \subset e(x_1, t_1) \cup \rho' \cup e(x_2, t_2)$$

connecting (x_1, t_1) to (x_2, t_2) . By definition of \sim , c is E -saturated, so $e(x_i, t_i) \subset c$. Thus $\rho \subset c$. Since

$$p_2(e(x_1, t_1) \cup \rho' \cup e(x_2, t_2)) \subset [t_1, t_2],$$

we have $p_2(\rho) \subset [t_1, t_2]$. \square

Proof of Theorem 5.21. Any $c \in \mathcal{T}_E$ is E -saturated by definition. Such c is connected (in fact, path connected) by Lemma 5.28. Thus \mathcal{T}_E satisfies (a). That it satisfies (b) follows from the definition of \sim and Remark 5.18. Condition (c) follows from Lemma 5.26. \square

Corollary 5.29. Let $s \in \mathcal{T}_E$.

- (a) s is closed from above.
- (b) For any $t \in p_2(s)$, s_t is connected.

Proof. By definition, s is a \sim -equivalence class. We rely on this in the following.

- (a) The claim is Lemma 5.23.
- (b) The claim is Lemma 5.24.

\square

Corollary 5.30. Let $s \in \mathcal{T}_E$ and let $s' \subset s$ be E -saturated. Then for any $a \subset p_2(s')$, we have

$$p_2^{-1}(a) \cap s = p_2^{-1}(a) \cap s'.$$

Proof. Let $t \in a$. Since s' is E -saturated, s'_t is an E -segment. So, Corollary 5.29(b) implies $s'_t = s_t$. \square

Corollary 5.31. Let $s \in \mathcal{T}_E$ and let $s' \subset s$ be E -saturated. Suppose $p_2(s')$ is connected. Then s' is connected.

Proof. Let $(x_i, t_i) \in s'$ for $i = 1, 2$. Suppose $t_1 \leq t_2$. Since $p_2(s')$ is connected, for any $t \in [t_1, t_2]$, $s'_t \neq \emptyset$. By Corollary 5.30 with $a = [t_1, t_2]$, we have

$$p_2^{-1}([t_1, t_2]) \cap s = p_2^{-1}([t_1, t_2]) \cap s'.$$

So, by Lemma 5.28 we can connect (x_1, t_1) to (x_2, t_2) by a path in s' . Since $(x_i, t_i) \in s'$ were arbitrary, s' is (path) connected. \square

Lemma 5.32. Let $c \in \mathcal{T}_E$. Let \mathcal{P} be a collection of disjoint connected E -saturated subsets of c . The relation \leq induces a linear order on \mathcal{P} .

Proof. First, we prove that for $P \subset \mathcal{P}$ we have

$$(29) \quad P \leq c_{T_f(c)}.$$

Indeed, for $t \in p_2(P)$ since P is E -saturated, Lemma 5.29(b) implies that $P_t = c_t$. By Lemma 5.10(b), we have $c_t \leq c_{T_f(c)}$. So,

$$P \leq P_t = c_t \leq c_{T_f(c)}$$

as desired.

Suppose $P_1, P_2 \in \mathcal{P}$. Without loss of generality, we may assume $P_1 \leq_2 P_2$. So, by relation (29) and Lemma 5.13, we have $P_1 \leq P_2$, which implies the lemma. \square

Lemma 5.33. *For each $s \in \mathcal{T}_E$ there exists a maximal element $m \in \mathcal{T}_E$ such that $s \leq m$.*

Proof. Let $x \in \overline{p_1(s)}$ be the point where f obtains its maximum. Then $f(x) \geq T_f(s)$. If $f(x) = T_f(s)$ then s itself is maximal. If $f(x) > T_f(s)$ we claim that $s \leq \{(x, f(x))\}$. Assume by contradiction otherwise. Then $x \notin p_1(s_{T_f(s)})$. By Lemma 5.29(b) $s_{T_f(s)}$ is a single E -segment. Therefore $p_1(s_{T_f(s)})$ is closed. It follows that x has an open neighborhood $v \subset \gamma \setminus p_1(s_{T_f(s)})$. Since $x \in \overline{p_1(s)}$ and f is continuous, there is a point $x' \in p_1(s) \cap v$ close enough to x so that $f(x') > T_f(s)$. Therefore $s \leq \{(x', f(x'))\}$. On the other hand, since $f(x') > T_f(s)$, $(x', f(x')) \notin s$. But since $x' \notin p_1(s_{T_f(s)})$, $s_{T_f(s)} \not\leq \{(x', f(x'))\}$ in contradiction to 5.21(c).

Let $m \in \mathcal{T}_E$ be the element containing $(x, f(x))$. Then we have that $s \leq \{(x, f(x))\}$ and $m \leq \{(x, f(x))\}$. Therefore, by Lemma 5.13 $s \leq m$. \square

Given a finite collection V of connected E -saturated sets, we define a graph F_V as follow. V is the set of vertices of F . We connect the vertex v_1 to v_2 if $v_1 \leq v_2$ and there is no $v_3 \in V$ such that $v_1 \leq v_3 \leq v_2$. By virtue of Corollary 5.22, F_V has no cycles and so is a forest. Again by Corollary 5.22, each tree T in F_V has a unique minimal vertex r_T , which we designate as the root of T . Thus the leaves of F_V are the maximal vertices. Denote by $R(V) \subset V$ the roots of F_V , by $L(V) \subset V$ the leaves, and by $I(V) \subset V$ the vertices which are neither roots nor leaves. Denote by $E(V)$ the edges of F_V .

A finite forest F is called **stable** if any $v \in F$ which is not a leaf has at least two direct descendants. The proof of the following lemma is standard and we omit it.

Lemma 5.34. *Let F be a finite stable forest and let L be the number of its leaves. Then $|F| \leq 2L$.*

Lemma 5.35. *Let $M \subset \mathcal{T}_E$ be the set of maximal elements under \leq . \mathcal{T}_E is finite if and only if M is finite. Moreover, in that case $|\mathcal{T}_E| \leq 2|M|$.*

Proof. First, we let S be an anti-chain in \mathcal{T}_E and show that $|S| \leq |M|$. By Lemma 5.33, for each $s \in S$ there is at least one $m \in M$ such that $s \leq m$. Since the elements of S are pairwise incomparable, and since by Corollary 5.22 the order on \mathcal{T}_E is tree-like, for any element $m \in M$ there is at most one $s \in S$ such that $s \leq m$. Thus $|S| \leq |M|$.

It suffices to prove the bound for any finite subset $V \subset \mathcal{T}_E$. Given such V , by Theorem 5.21(c) we may choose $V' \subset \mathcal{T}_E$, such that $V \subset V'$ and $F_{V'}$ is stable. The claim now follows from Lemma 5.34. \square

6. THICK THIN PARTITION

6.1. Thickened hypograph. We now specialize the discussion of the previous section to the case where γ is a geodesic in Σ as in Theorem 4.2. Denote the connected components of γ by γ_i . We will assume throughout this section that for all i we have

$$(30) \quad 4e^{-\max_{x \in \gamma_i} g(x)} \leq \ell_n(\gamma_i),$$

and

$$(31) \quad 2k \leq \max_{x \in \gamma_i} e^{g(x)}.$$

Recall Definition 4.6. Let $s_t := \{c \in \pi_0(X_t \setminus D) \mid \ell_n(c) > 4e^{-t}\}$, and let S_t be the union of elements of s_t . Let $E_t := X_t \setminus S_t$, and $E := \cup_t E_t \cup \gamma \times \{\ln 2k\}$. We will show that E is the hypograph of a continuous function. Figure 8 gives a picture of a typical E compared with D .

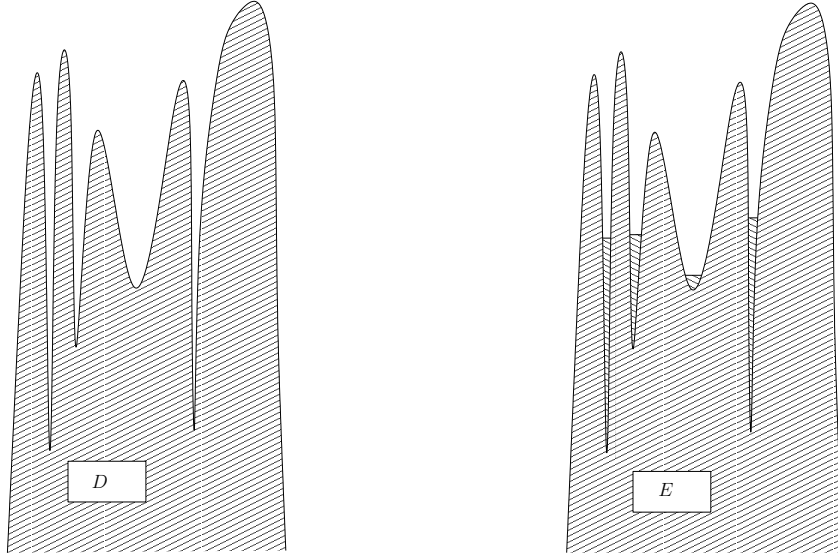


FIGURE 8.

Remark 6.1. By definition D_t is $2e^{-t}$ -dense in E_t . Let $t_1, t_2 \geq \ln(2k)$. Let $S_1 \subset E_{t_1}, S_2 \subset E_{t_2}$, be segments such that $p_1(S_1) \cap p_1(S_2) = \emptyset$ and such that $\ell_n(S_i) \geq 8e^{-t_i}$ for $i = 1, 2$. Then S_i contains a point $x_i \in D_t$ such that $d(x_i, \partial S_i; h_n) \geq 2e^{-t_i}$. Let $B_i = B(x_i, t_i)$. By Lemma 4.7, $\mu(B_i) \geq \delta_1$. Furthermore, by Lemma 4.8

$$B_1 \cap B_2 = \emptyset.$$

This observation will play a key role in the following.

Lemma 6.2. *E is closed.*

Proof. Let $(x, t) \in \partial E$. We show that $(x, t) \in E$. If $(x, t) \in D$, we are done since $D \subset E$. Otherwise, let s be the connected component of $X_t \setminus D$ containing (x, t) . We show that $\ell_n(s) \leq 4e^{-t}$, which implies that $s \subset E$. Assume by contradiction otherwise. Then there is a compact segment $s' \subset s$ such that $\ell_n(s') > 4e^{-t}$ and such that $(x, t) \in s'^o$. By the closedness of D , compactness of s' , and continuity of the exponent, there is an $\epsilon > 0$ such that $\pi_1(s') \times (t - \epsilon, t + \epsilon) \subset X \setminus D$ and such that for any $t' \in (t - \epsilon, t + \epsilon)$, $\ell_n(s') > 4e^{-t'}$. This implies that $\pi_1(s') \times (t - \epsilon, t + \epsilon) \subset X \setminus E$. But $\pi_1(s') \times (t - \epsilon, t + \epsilon)$ contains an open neighborhood of (x, t) in contradiction to the fact that $(x, t) \in \partial E$. \square

Lemma 6.3. *Let $t \in [\ln 2k, \infty)$. For all $x \in p_1(E_t)$, $\{x\} \times [\ln 2k, t) \subset E^o$.*

Proof. First, we prove that for all $y \in p_1(E_t)$,

$$(32) \quad \{y\} \times [\ln 2k, t) \subset E.$$

Let $t' \in [\ln 2k, t)$. We show that $(y, t') \in E$. Indeed, if $(y, t') \in D$, we are done since $D \subset E$. If $(y, t') \notin D$, then $(y, t) \notin D$. Let S and S' be the components of $X_t \setminus D$ and $X_{t'} \setminus D$ containing (y, t) and (y, t') respectively. Clearly $p_1(S') \subset p_1(S)$. Since $(y, t) \in E$, we have by definition of E that

$$\ell_n(S') \leq \ell_n(S) \leq 4e^{-t} < 4e^{-t'}.$$

By the same definition, we conclude that $(y, t') \in E$.

To prove the claim, it suffices to show for any $t' \in [\ln 2k, t)$ that $(x, t') \in E^o$. If $t' < g(x)$, then $(x, t') \in D^o \subset E^o$ by continuity of g . If $t' \geq g(x)$, then $t > g(x)$. Let S be the component of $X_t \setminus D$ containing (x, t) . By definition of E , we have $S \subset E_t$. So, invoking inclusion (32) for each $y \in \pi_1(S)$, we conclude that the open neighborhood $\pi_1(S) \times [\ln 2k, t)$ of (x, t') is contained in E . \square

It is clear from the definition that for any $x \in \gamma$, the set

$$\{t \in [\ln 2k, \infty) \mid (x, t) \in E\}$$

is bounded from above. It follows from Lemma 6.3 that E is the hypograph of the function $f_{\partial E}$ defined by

$$f_{\partial E}(x) = \sup\{t \in [\ln 2k, \infty) \mid (x, t) \in E\}.$$

Lemma 6.4. *∂E is the graph of $f_{\partial E}$.*

Proof. Clearly, the graph of f is contained in ∂E . So, it suffices to prove the opposite inclusion. Let $(x, t) \in \partial E$. Then by Lemma 6.2 $(x, t) \in E$, so by Lemma 6.3 $\{x\} \times [\ln 2k, t) \subset E^\circ$. On the other hand, we show that $\{x\} \times (t, \infty) \subset X \setminus E$. Indeed, if $t' \in (t, \infty)$ and $(x, t') \in E$, then by Lemma 6.3

$$(x, t) \in \{x\} \times [\ln 2k, t') \subset E^\circ$$

contradicting the choice of (x, t) . Thus, by definition $f_{\partial E}(x) = t$. \square

Corollary 6.5. $f_{\partial E}$ is continuous.

Proof. The graph of $f_{\partial E}$ is closed being the boundary of E . $f_{\partial E}$ is bounded, so its graph is compact. Thus, the projection p_1 restricted to the graph of $f_{\partial E}$ is closed. It follows that $f_{\partial E}$ is continuous. \square

Remark 6.6. It follows from equations (30) and (31) that

$$\max_{x \in \gamma} f_{\partial E}(x) = \max_{x \in \gamma} g(x).$$

Lemma 6.7. Let e_1 and e_2 be E -segments that are incomparable with respect to \leq . Let $t_i = p_2(e_i)$. Then,

$$d(p_1(e_1), p_1(e_2); h_n) \geq 4e^{-\min(t_1, t_2)}.$$

Proof. Assume without loss of generality that $t_1 \leq t_2$. Let $x \in p_1(e_2)$ and denote $e'_2 = e(x, t_1)$. Since e'_2 and e_1 are both connected components of E_{t_1} , we have by the definition of E_{t_1} that either $e'_2 = e_1$ or $d(p_1(e'_2), p_1(e_1)) \geq 4e^{-t_1}$. By Lemma 5.3(c), $e'_2 \leq e_2$. Thus $p_1(e_2) \subset p_1(e'_2)$ giving the claim. \square

Lemma 6.8. Let $S_1, S_2 \subset E$. Suppose S_1 is E -saturated and connected, S_2 is connected and $S_1 \leq_2 S_2$. Then either $S_1 \leq S_2$ or

$$d(p_1(S_1), p_1(S_2); h_n) \geq 4e^{-T_i(S_1)}.$$

Proof. Suppose $S_1 \not\leq S_2$. Let $(x_1, t_1) \in S_1$ and $(x_2, t_2) \in S_2$. Let $e_i = e(x_i, t_i)$ for $i = 1, 2$. We show that $d(x_1, x_2) \geq 4e^{-T_i(S_1)}$. Suppose $t \in p_2(S_1)$ is such that $t \leq t_1$. Since S_1 is E -saturated and connected, Lemma 5.10(b) implies that $(x_1, t) \in S_1$. Since $S_1 \leq_2 S_2$, there is a $t \in p_2(S_1)$ such that $t \leq t_2$. We may thus assume without loss of generality that $t_1 \leq t_2$. Furthermore, t_1 may be assumed to be arbitrarily close to $T_i(S_1)$. Since S_1 is E -saturated, $e_1 \subset S_1$. By Lemma 5.12 we have $p_1(S_1) \cap p_1(S_2) = \emptyset$. In particular $x_2 \notin p_1(e_1)$, so $e_1 \not\leq e_2$. Since $t_1 \leq t_2$, e_1 and e_2 are incomparable. Therefore, by Lemma 6.7 we have

$$d(x_1, x_2) \geq d(p_1(e_1), p_1(e_2)) \geq 4e^{-t_1}.$$

Since t_1 is arbitrarily close to $T_i(S_1)$, we have

$$d(x_1, x_2) \geq 4e^{-T_i(S_1)}.$$

Since x_i were arbitrary points in $p_1(S_i)$, the claim follows. \square

Corollary 6.9. *Let $S_1, S_2 \subset E$ be incomparable connected E -saturated sets. Then*

$$d(p_1(S_1), p_1(S_2); h_n) \geq 4e^{-\min\{T_i(S_1), T_i(S_2)\}}.$$

Proof. Without loss of generality $S_1 \leq_2 S_2$. The claim thus follows from Lemma 6.8. \square

Corollary 6.10. \mathcal{T}_E is finite.

Proof. Let M be the set of maximal elements of \mathcal{T}_E . Then the elements of M are pairwise incomparable under \leq . Let $T = \sup_{x \in \gamma} g(x)$. By Remark 6.6 we have $T_i(c) \leq T$ for any $c \in M$. It follows from Corollary 6.9 that

$$|M| \leq \frac{1}{4}e^T \ell_n(\gamma).$$

In particular, M is finite. The claim therefore follows from Lemma 5.35. \square

6.2. Thick thin partition. We now wish to partition E into a thin part and a thick part. To this end, let

$$N := \{(x, t) \in E \mid \ell_n(e(x, t)) \leq 24e^{-t}\},$$

and let

$$K := E \setminus N.$$

Lemma 6.11. N is E -saturated and closed from above.

Proof. It is obvious that N is E -saturated. We show that N is closed from above. Let (x, t) be a right boundary point of $\{x\} \times [\ln 2k, \infty) \cap N$. Since E is closed, we have $(x, t) \in E$. Thus we need to show that $\ell_n(e(x, t)) \leq 24e^{-t}$. Assume by contradiction otherwise. Let $\epsilon > 0$ be so small that $\ell_n(e(x, t)) > 24e^{\epsilon-t}$. By Lemma 6.3, we have

$$\{x\} \times [t - \epsilon, t) \subset E.$$

On the other hand, for any $t' \in [t - \epsilon, t)$, by Lemma 5.10(c)

$$p_1(e(x, t)) \subset p_1(e(x, t')).$$

So,

$$\ell_n(e(x, t')) \geq \ell_n(e(x, t)) > 24e^{-t+\epsilon} \geq 24e^{-t'}.$$

Therefore, $\{x\} \times [t - \epsilon, t) \subset E \setminus N$ giving a contradiction. \square

Definition 6.12. A **thin neck** is a connected component of $c \cap N$ where $c \in \mathcal{T}_E$. Given a thin neck L , we denote by c_L the unique $c \in \mathcal{T}_E$ such that $L \subset c$.

Lemma 6.13. *Let L be a thin neck.*

- (a) *L is E -saturated and closed from above.*
- (b) *Let $S \subset E$ be disjoint from L , then*

$$L \leq S \Rightarrow L_{T_f(L)} \leq S.$$

- (c) *For any $x \in p_1(L) \setminus p_1(L_{T_f(L)})$,*

$$(x, f_{\partial E}(x)) \in L.$$

- (d) *For any $x \in p_1(L) \setminus p_1(L_{T_f(L)})$,*

$$g(x) \leq \ln 24 - \ln d(x, p_1(L_{T_f(L)}); h_n).$$

Proof. (a) Let $c \in \mathcal{T}_E$ such that L is a connected component of $c \cap N$. c is E saturated by definition and closed from above by Corollary 5.29(a). N is E -saturated and closed from above by Lemma 6.11. It follows from Remark 5.5 that L is E -saturated. By Remark 5.20, L is closed from above.

(b) L is connected and contained in an equivalence class $c \in \mathcal{T}_E$ by definition. L is E -saturated and closed from above by (a). The claim thus follows from Theorem 5.21(c).

(c) Let $t \geq \ln 2k$ be such that $(x, t) \in L$. Since $L \subset E$, we have $t \leq f_{\partial E}(x)$. This means that $L \leq \{(x, f_{\partial E}(x))\}$. Suppose

$$(x, f_{\partial E}(x)) \notin L.$$

Then by (b), $L_{T_f(L)} \leq \{(x, f_{\partial E}(x))\}$. This produces the contradiction $x \in p_1(L_{T_f(L)})$.

- (d) Let $x' \in p_1(L_{T_f(L)})$. By (c), $(x, f_{\partial E}(x)) \in L$, so $f_{\partial E}(x) \in p_2(L)$. Thus by 5.10(b) we have $x' \in p_1(L_{f_{\partial E}(x)})$. Let $c \in \mathcal{T}_E$ be such that $L \subset c$. By (a), L is E -saturated, so by Corollary 5.29 we have $L_{f_{\partial E}(x)} = c_{f_{\partial E}(x)}$ and $L_{f_{\partial E}(x)}$ is connected. Thus $e(x, f_{\partial E}(x)) = e(x', f_{\partial E}(x))$. But

$$\ell_n(e(x', f_{\partial E}(x))) \leq 24e^{-f_{\partial E}(x)},$$

so

$$\begin{aligned} (33) \quad d(x, p_1(L_{T_f(L)}); h_n) &\leq d(x, x'; h_n) \\ &\leq \ell_n(e(x', f_{\partial E}(x))) \\ &\leq 24e^{-f_{\partial E}(x)} \\ &\leq 24e^{-g(x)}. \end{aligned}$$

The claim follows by taking logarithms in (33). \square

Lemma 6.14. *For $i = 1, 2$, let L_i be disjoint thin necks such that $p_2(L_1) \cup p_2(L_2)$ is connected.*

- (a) $c_{L_1} \neq c_{L_2}$.
- (b) If $L_1 \leq L_2$, then $L_{1, T_f(L_1)}$ contains a branching point.

Proof. (a) Suppose the contrary. Then Corollary 5.31 implies that $L_1 \cup L_2$ is connected. The definition of thin necks implies the contradiction $L_1 = L_2$.

- (b) Let $(x, t) \in L_2$. For any $t' \in [T_f(L_1), t)$, denote

$$R(t') = p_1(e(x, t')) \times [t', t).$$

Since $L_1 \leq L_2$, by Lemma 6.13(b), $(x, T_f(L_1)) \in L_1$. By (a), $c_{L_1} \neq c_{L_2}$. Therefore, $(x, t) \not\sim (x, T_f(L_1))$. Thus $R(T_f(L_1))$ contains a branching point p . But for any $t' \in (T_f(L_1), t)$, by Lemma 5.27, $\{x\} \times [t', t] \subset c_{L_2}$. In particular, $(x, t) \sim (x, t')$. So, $R(t')$ contains no branching point. Thus $p \in e(x, T_f(L_1))$. Since $(x, T_f(L_1)) \in L_1$, by Lemma 6.13(a), $e(x, T_f(L_1)) \subset L_{1, T_f(L_1)}$. Therefore, $p \in L_{1, T_f(L_1)}$. \square

Definition 6.15. Let L be a thin neck. L is **exceptional** if it satisfies the following two conditions.

- (a) $T(L) < \ln 3$.
- (b) For any $\epsilon > 0$ there are $(x_i, t_i) \in K$ for $i = 1, 2$, such that $t_1 \in (T_i(L) - \epsilon, T_i(L)]$, $t_2 \in (T_f(L), T_f(L) + \epsilon)$, and $e(x_1, t_1) \leq L \leq e(x_2, t_2)$.

Definition 6.16. Let L be a thin neck. We write

$$\mathcal{N}^+(L, \epsilon) := (p_1(L) \times [T_i(L), T_f(L) + \epsilon)) \cap E,$$

and

$$\mathcal{N}^-(L, \epsilon) := (p_1(L) \times [T_i(L) - \epsilon, T_f(L))) \cap E.$$

We say that L is **upper interior** if there is an $\epsilon > 0$ such that $\mathcal{N}^+(L, \epsilon) \subset N$. Similarly, we say that L is **lower interior** if there is an $\epsilon > 0$ such that $\mathcal{N}^-(L, \epsilon) \subset N$.

Remark 6.17. By Definition 6.15(b), every non-exceptional thin neck L such that $T(L) < \ln 3$ is either upper interior or lower interior.

Remark 6.18. It is clear that for any $\epsilon > 0$, $\mathcal{N}^+(L, \epsilon)$ and $\mathcal{N}^-(L, \epsilon)$ are connected.

Definition 6.19. The **thin part** $A \subset E$ is the union of non-exceptional thin necks. The **thick part** of E is

$$C := E \setminus A.$$

Lemma 6.20. C is E -saturated.

Proof. This follows from Lemma 6.13(a) and Remark 5.5. \square

6.3. Energy bound on the number of thin necks. Let H denote the set of non-exceptional thin necks L such that $T(L) < \ln 3$. Let G denote the set of all non-exceptional thin necks.

Lemma 6.21.

$$|H| \leq 2|\mathcal{T}_E|.$$

Proof. We will show that the map $H \rightarrow \mathcal{T}_E$ defined by $L \mapsto c_L$ is at most two to one. Let $c \in \mathcal{T}_E$. By Remark 6.17 every $L \in H$ is either upper interior or lower interior. We claim that there is at most one lower interior $L \in H \cap \pi_0(N \cap c)$. Similarly we claim that there is at most one upper interior $L \in H \cap \pi_0(N \cap c)$.

Indeed, suppose $L \subset c$ is a lower interior element of H . First, we show that

$$(34) \quad T_i(L) = T_i(c).$$

Since $L \subset c$, we have $T_i(c) \leq T_i(L)$. Suppose by contradiction the inequality is strict. Let $\epsilon > 0$ be such that $\mathcal{N}^-(L, \epsilon) \subset N$. By Theorem 5.21(a), c is connected, so $p_2(c)$ is an interval. Thus we may choose $t_1 \in p_2(c) \cap [T_i(L) - \epsilon, T_i(L))$. Choose $(x_2, t_2) \in L$. By Lemma 5.27, $\{x_2\} \times [t_1, t_2] \subset c$. By definition of $\mathcal{N}^-(L, \epsilon)$, we have $\{x_2\} \times [t_1, t_2] \subset \mathcal{N}^-(L, \epsilon) \subset N$. It follows that (x_2, t_1) belongs to the same connected component of $c \cap N$ as (x_2, t_2) . However $t_1 < T_i(L)$, so $(x_1, t_1) \notin L$ contradicting the definition of L . Equation (34) follows.

Let $L' \in H \cap \pi_0(N \cap c)$ also be lower interior. Then $T_i(L') = T_i(c) = T_i(L)$. In particular, $p_2(L) \cup p_2(L')$ is connected. Therefore, by Lemma 6.14(a), $L = L'$.

The upper interior case follows similarly. Thus the map $L \mapsto c_L$ is at most two to one. \square

Corollary 6.22. $|G| < \infty$.

Proof. In light of Lemmas 6.21 and 6.10 we need only bound $G \setminus H$. Again relying on Lemma 6.10, it suffices to bound the set S_c defined for any $c \in \mathcal{T}_E$ by

$$S_c = (G \setminus H) \cap \pi_0(N \cap c).$$

By Remark 6.6 we have

$$T(c) \leq \sup_{x \in \gamma} g(x) - \ln 2k =: M.$$

Lemma 6.14(a) implies that $p_2(L_1) \cap p_2(L_2) = \emptyset$ for any two elements $L_1, L_2 \in S_c$. Therefore $|S_c| \leq \frac{T(c)}{\ln 3} \leq \frac{M}{\ln 3}$. \square

Lemma 6.23. *Let $c \in \mathcal{T}_E$ and let $L_1, L_2 \subset c$ be non-exceptional thin necks. Let k be a connected component of $c \cap C$ satisfying $L_1 \leq k \leq L_2$. Then k is a connected component of C .*

Proof. By Remark 5.5 and Lemma 6.20, k is E -saturated. Choose $(x, t_2) \in L_2$. Since $L_1 \leq k \leq L_2$, there exist $t_1 \in p_2(L_1)$ and $t \in p_2(k)$ such that $(x, t_1) \in L_1$ and $(x, t) \in k$. Since $L_1, L_2 \subset A$ and $k \subset C$, we have

$$L_i \cap k = \emptyset, \quad i = 1, 2.$$

So, by Lemma 5.9 and Corollary 5.11, we deduce that $t_1 < t < t_2$.

Let k' be the connected component of C containing k . We prove that $k' \subset c$, which immediately implies the lemma. Indeed, let

$$R = p_1(e(x, t_1)) \times [t_1, t_2].$$

Since $L_1, L_2 \subset c$, we have $(x, t_1) \sim (x, t_2)$. So, $e(x, t_1) \leq e(x, t_2)$ and R contains no branch points. Since $e(x, t_1) \leq e(x, t_2)$, we have

$$e(x, t_2) \subset (\overline{R} \cap E)_{t_2}.$$

Since R contains no branched points, by Lemma 5.17 we deduce that $(\overline{R} \cap E)_{t_2}$ is connected. So,

$$e(x, t_2) = (\overline{R} \cap E)_{t_2}.$$

Since $L_i \subset A$ does not intersect C ,

$$e(x, t_i) \subset L_i$$

does not intersect C for $i = 1, 2$. Let x_1 and x_2 be the endpoints of the interval $p_1(e(x, t_1))$. By Remark 5.1,

$$(\{x_i\} \times (t_1, \infty)) \cap C \subset (\{x_i\} \times (t_1, \infty)) \cap E = \emptyset.$$

It follows that $\partial R \cap C = \emptyset$. So, $C_1 = C \setminus \overline{R}$ and $C_2 = C \cap R^\circ$ constitute a partition of C into relatively open subsets. Since $(x, t) \in R^\circ$ and $(x, t) \in k \subset k'$, the connectedness of k' implies that $k' \subset C_1$. Therefore, by the definition of \sim , we conclude that $k' \subset c$ as desired. The lemma follows. \square

Lemma 6.24. *Let L be an exceptional thin neck. Then $L_{T_f(L)}$ does not contain a branching point.*

Proof. Let $R = p_1(L_{T_f(L)}) \times (T_f(L), \infty) \cap E$, and let $\epsilon > 0$. By Definition 6.15(b) there exists $t \in (T_f(L), T_f(L) + \epsilon)$ and a component k of R_t satisfying $k \subset K$. Suppose now that $L_{T_f(L)}$ contains a branching point. Then by Remark 5.18 there exists $t' \in (T_f(L), t]$ such that $R_{t'}$ has at least two components. Therefore, there is at least one component e of $R_{t'}$ such that $e \not\subset k$. By Lemma 6.8, $d(p_1(e), p_1(k)) \geq 4e^{-t'}$. Furthermore, $L_{T_f(L)} \leq e \cup k$. So, since $k \subset K$,

$$\begin{aligned}
 (35) \quad \ell_n(p_1(L_{T_f(L)})) &> \ell_n(p_1(e)) + \ell_n(p_1(k)) + 4e^{-t'} \\
 &> 24e^{-t} + 4e^{-t'} \\
 &> 28e^{-(T_f(L)+\epsilon)}.
 \end{aligned}$$

On the other hand, since $L \subset N$, $\ell_n(L_{T_f(L)}) \leq 24e^{-T_f(L)}$. This together with Equation (35) implies that

$$28e^{-(T_f(L)+\epsilon)} < 24e^{-T_f(L)}.$$

Since ϵ is arbitrary, we obtain a contradiction. \square

Corollary 6.25. *Let L be an exceptional thin neck. Let L' be any thin neck such that $L \not\leq L'$. Then $T_i(L') > T_f(L)$.*

Proof. Suppose the contrary. Then by Lemma 6.14(b), $L_{T_f(L)}$ contains a branching point. This contradicts Lemma 6.24. \square

Corollary 6.26. *Let L be an exceptional thin neck. There is an $\epsilon > 0$ such that $\mathcal{N}^+(L, \epsilon) \subset C$.*

Proof. Let S be the set of non-exceptional thin necks L' such that $L \leq L'$. It follows from Corollary 6.22 that S is finite. Let $t = \min_{s \in S} T_i(s)$. By Corollary 6.25, $t > T_f(L)$. Let $\epsilon = t - T_f(L)$. Then ϵ satisfies the requirement. \square

Lemma 6.27. *Let $t \in p_2(C)$ and let c be a connected component of C_t . There is a $t' \in [t, t + \ln 3)$ and a component c' of $K_{t'}$ such that $c \leq c'$. Furthermore, c' can be taken to belong to the same connected component of C as c .*

Proof. If $c \subset K$, take $t' = t$. Otherwise, since by Lemma 6.20, c is an E -segment, and by Remark 5.5 K is E -saturated, we have $c \cap K = \emptyset$. So, $c \subset N \cap C$. Therefore, c is contained in an exceptional thin neck L . Using Corollary 6.26, choose ϵ_1 small enough that $\mathcal{N}^+(L, \epsilon_1) \subset C$. Let

$$0 < \epsilon < \min(\ln 3 - (T_f(L) - t), \epsilon_1).$$

By Definition 6.15(b) and Lemma 6.13(b), there is a

$$t' \in (T_f(L), T_f(L) + \epsilon)$$

and an

$$x \in p_1(K_{t'})$$

such that $L_{T_f(L)} \leq e(x, t')$. By Lemma 6.13(a), again using the fact that c is an E -segment, $c = L_t$. So, by Lemma 5.10(b)

$$c \leq L_{T_f(L)} \leq e(x, t').$$

By the choice of ϵ ,

$$t < t' < T_f(L) + \epsilon < \ln 3 + t.$$

So, we take $c' = e(x, t')$. By Remark 6.18, $\mathcal{N}^+(L, \epsilon_1)$ is a connected subset of C that contains both c and c' . \square

Definition 6.28. A subset $Z \subset \Sigma$ is **dense** if $\mu(Z) \geq \delta_1$.

Definition 6.29. Let S be a set. An **energy partition** for S is a map which assigns to each element $e \in S$ a dense subset $Z(e) \subset \Sigma$ in such a way that if $e_1 \neq e_2 \in S$, $Z(e_1) \cap Z(e_2) = \emptyset$.

Remark 6.30. Any set S that carries an energy partition satisfies $|S| \leq \frac{E}{\delta_1}$. Let F be a finite forest, and denote by Y_F the set of vertices of F with at most one child. It is easy to see that $|F| \leq 2|Y_F|$. So, if Y_F admits an energy partition, then

$$|F| \leq 2 \frac{\mu\{\Sigma\}}{\delta_1}.$$

Lemma 6.31. $|\mathcal{T}_E| \leq 2 \frac{\mu\{\Sigma\}}{\delta_1}$.

Proof. Let $c \in \mathcal{T}_E$ and let $t_c \in p_2(c)$. By definition of E and Remark 6.6, we have $c_{t_c} \cap D_{t_c} \neq \emptyset$. Let $x_c \in c_{t_c} \cap D_{t_c}$ and let $B_c = B(x_c, t_c)$. By Lemma 4.7, $\mu\{B_c\} \geq \delta_1$. Let $c_1 \neq c_2 \in \mathcal{T}_E$ be maximal elements. By Corollary 6.9

$$d(x_{c_1}, x_{c_2}) > 2(e^{-t_{c_1}} + e^{-t_{c_2}}).$$

Thus, by Lemma 4.8, $B_{c_1} \cap B_{c_2} = \emptyset$. The claim follows from Remark 6.30 and Lemma 5.35. \square

Lemma 6.32. $|\pi_0(C)| \leq 10 \frac{\mu\{\Sigma\}}{\delta_1}$.

Proof. Let $P \subset \pi_0(C)$ be finite. Let Q be the set of vertices of F_P with at most one child. We partition Q into two sets, Q_1 and Q_2 , such that Q_1 carries an energy partition, and Q_2 is mapped two to one into \mathcal{T}_E . Let $Q' \subset Q$ be the set of vertices with exactly one child. Let $q \in Q'$, let q' be the unique child of q , and let $(x, t') \in q'$. Since $q \leq q'$, there exists $t \in p_2(q)$ such that $(x, t) \in q$. By Lemma 5.9, we have $t < t'$. Let

$$L_q = \{x\} \times [t, t'].$$

Since q and q' are different components of C , and L_q is a path connecting them, it follows that L_q intersects at least one non-exceptional thin neck. So, for each $q \in Q'$, choose a non-exceptional thin neck g_q such that $g_q \cap L_q \neq \emptyset$. Denote by Q_2 the subset of elements $q \in Q'$ for which $g_q \in H$. By Lemmas 6.21 and 6.31, $|Q_2| \leq 4\frac{\mu\{\Sigma\}}{\delta_1}$.

Denote $Q_1 = Q \setminus Q_2$. We define an energy partition for Q_1 as follows. To each $q \in Q_1$ we associate a $t_q \in p_2(q)$ and a connected segment $s_q \subset q_{t_q}$ in such a way that the following conditions hold:

(a)

$$\ell_n(s_q) \geq 8e^{-t_q}.$$

(b)

$$p_1(s_q) \times [t_q, \infty) \cap (\cup_{\{q'' \in Q_1 | q \leq q''\}} q'') = \emptyset.$$

Indeed, suppose $q \in Q_1$ has no children. By Lemma 6.27 there is a $t_q \in p_2(q)$ such that q_{t_q} contains a component of K_{t_q} . Let s_q be a connected component of $K_{t_q} \cap q_{t_q}$. Such s_q satisfies condition (a) by definition of K and condition (b) because q has no descendants.

Otherwise $q \in Q'$. Let q', x, t, t' and g_q be as above. By Lemma 6.27 we may choose $t_q \in p_2(q)$ and a connected component $k_q \subset q_{t_q} \cap K$ such that $e(x, t) \leq k_q$. In particular, $(x, t_q) \in k_q$. By choice of g_q , there exists $t_{g_q} \in p_2(g_q)$ such that $(x, t_{g_q}) \in L_q$. Since $q, q' \subset C$, we have

$$(36) \quad q \cap g_q = \emptyset = q' \cap g_q.$$

In particular, $t' > t_{g_q} > t$. It follows from Lemma 5.9 that

$$(37) \quad q \leq g_q \leq q'.$$

By Lemma 6.13(b), we have

$$(38) \quad (g_q)_{T_f(g_q)} \leq q'.$$

We show that

$$(39) \quad t_{g_q} > t_q.$$

Indeed, if $t_{g_q} < t_q$, then Lemma 5.9 would imply that $g_q \leq q$, which in light of equation (36) and relation (37), contradicts Corollary 5.11. Also, $t_{g_q} \neq t_q$ by equation (36). Inequality (39) follows. Lemma 5.9 and inequality (39) imply that $k_q \leq g_q$. So,

$$(40) \quad t_q \leq T_i(g_q).$$

Since $q \in Q_1$, we have $g_q \in G \setminus H$, that is $T(g_q) \geq \ln 3$. So, by inequality (40), we have

$$(41) \quad T_f(g_q) \geq T_i(g_q) + \ln 3 \geq t_q + \ln 3.$$

By definition of K and N respectively, we have

$$\ell_n(p_1(k_q)) > 24e^{-t_q}, \quad \ell_n(p_1((g_q)_{T_f(g_q)})) \leq 24e^{-T_f(g_q)}.$$

So, by relation (38) and inequality (41), we conclude that

$$\ell_n(p_1(q')) \leq \ell_n(p_1((g_q)_{T_f(g_q)})) \leq 24e^{-(\ln 3 + t_q)} < \frac{1}{3}\ell_n(p_1(k_q)).$$

Therefore, there exists an $s_q \subset k_q \subset q_{t_q}$ such that $\ell_n(s_q) \geq 8e^{-t_q}$ and such that $p_1(q') \cap p_1(s_q) = \emptyset$. By construction s_q satisfies condition (a). We show it satisfies condition (b) as follows. Since q' is the unique child of q , any $q'' \in Q_1 \setminus \{q\}$ such that $q \leq q''$ must satisfy $q' \leq q''$. In particular, $p_1(q'') \subset p_1(q')$. Thus $p_1(q'') \cap p_1(s_q) = \emptyset$ implying condition (b).

We claim that if $q \neq q' \in Q_1$, then

$$(42) \quad p_1(s_q) \cap p_1(s_{q'}) = \emptyset.$$

Indeed, if $q \neq q' \in Q_1$ are comparable, condition (b) implies equation (42). So, suppose they are incomparable. We may assume without loss of generality that $q \leq_2 q'$. Thus since $s_q \subset q$, equation (42) follows from Lemma 5.12. By condition (a), equation (42) and Remark 6.1, we can associate with each q a dense disk B_q such that if $q \neq q'$, then $B_q \cap B_{q'} = \emptyset$. The assignment $q \mapsto B_q$ is an energy partition.

Therefore, we have

$$|P| \leq 2(|Q_1| + |Q_2|) \leq 10 \frac{\mu(\Sigma)}{\delta_1}.$$

Since P was an arbitrary finite subset, it follows that $\pi_0(C)$ itself is finite and thus satisfies the same inequality. \square

For $c \in \mathcal{T}_E$, let \mathcal{P}_c denote the partition of c into the connected components of $c \cap C$ together with the connected components of $c \cap A$. That is, \mathcal{P}_c is the partition of c into non-exceptional thin necks and the connected components of their complement.

Lemma 6.33. *Let $c \in \mathcal{T}_E$.*

- (a) *The set \mathcal{P}_c is well-ordered under the relation \leq and finite.*
- (b) *If $L \in \mathcal{P}_c$ is a non-exceptional thin neck that has a successor, its successor is a component of $c \cap C$.*

Proof. (a) By Corollary 6.22, the set of non-exceptional thin necks in \mathcal{P}_c is finite. We denote them by L_1, \dots, L_n . Let $a_i = p_2(L_i)$ for $i = 1, \dots, n$. Since L_i is connected, a_i is an interval. Since c

is connected, $p_2(c)$ is an interval. So, $p_2(c) \setminus (L_1 \cup \dots \cup L_n)$ is a finite collection of intervals b_1, \dots, b_k , with $k \leq n + 1$. Let

$$M_j = p_2^{-1}(b_j) \cap c$$

for $j = 1, \dots, k$. By definition, M_j is E -saturated. By Corollary 5.31, M_j is connected. By Corollary 5.30,

$$c = \bigcup_{i=1}^n L_i \cup \bigcup_{j=1}^k M_j.$$

By the construction of the M_j , the union is disjoint. Therefore,

$$\mathcal{P}_c = \{L_1, \dots, L_n, M_1, \dots, M_k\}.$$

In particular, \mathcal{P}_c is finite. By Lemma 5.32, it is well-ordered.

- (b) We continue using the notation of part (a). Without loss of generality, we may assume $L_i \leq L_{i+1}$ for $i = 1, \dots, n - 1$. Let i be such that $L = L_i$. If the successor of L_i were not a component of $c \cap C$, it would have to be L_{i+1} . Assume so by way of contradiction. But the union $a_i \cup a_{i+1}$ cannot be connected by Lemma 6.14(a). So, there is a j such that $a_i \leq b_j \leq a_{i+1}$, where \leq denotes the usual order on \mathbb{R} . It follows that $L_i \leq_2 M_j \leq_2 L_{i+1}$. By Lemma 5.32, we conclude that $L_i \leq M_j \leq L_{i+1}$, contradicting the assumption that L_{i+1} is the successor of L_i .

□

Lemma 6.34. $|G| \leq 12 \frac{\mu\{\Sigma\}}{\delta_1}$.

Proof. It suffices to define an injective map $i : G \rightarrow \pi_0(C) \coprod \mathcal{T}_E$. Let $L \in G$. If L is maximal in the set of thin necks which are components of c_L , map L to c_L . Otherwise, L is not a maximal element of \mathcal{P}_{c_L} . By Lemma 6.33(a), \mathcal{P}_{c_L} is well-ordered. So, we let $j_L \in \mathcal{P}_{c_L}$ be the successor of L . By Lemma 6.33(b), we have $j_L \subset c_L \cap C$. Since L is not a maximal thin-neck in c_L , Lemma 6.23 implies that j_L is a connected component of C . So, we map L to j_L .

We prove that i is injective. Indeed, suppose $i(L) = c_L \in \mathcal{T}_E$. By Lemma 5.32 maximal thin necks in c_L are unique. So, there can be no other thin neck mapped to c_L . On the other hand, suppose $i(L) = j_L \in \pi_0(C)$. By construction j_L is a subset of a unique $c \in \mathcal{T}_E$. It has a unique predecessor in \mathcal{P}_c , which is L . So, no other thin neck can be mapped to j_L .

□

Remark 6.35. It is straightforward to verify that all constructions of this section, namely E , N , K , C , A and G , are conjugation invariant.

7. TAME GEODESICS

Definition 7.1. Let $k > 0$ and let I be a cylinder. A compact embedded geodesic $\gamma \subset I$ is said to be k -tame if for any sub-cylinder $I' \subset I$, we have

$$\ell_{h_{st}}(\gamma \cap I') \leq 2\pi k \max(\text{Mod} I', 1).$$

Lemma 7.2. *There are constants a_1, a_2 such that the following holds. Let $k_1, k_2 > 0$ and let $(\Sigma, j, \mu) \in \mathcal{M}$. Let $I \subset \Sigma_{\mathbb{C}}$ be clean and doubly connected. Let $\gamma : [0, 1] \rightarrow I$ be k_1 -tame. Let μ be a thick thin measure on I and let*

$$f(x) := \min \left(\frac{d\ell_{\mu}}{d\ell_{h_{st}}}, k_2 \right).$$

Then

$$\int_0^1 f(\gamma(t)) \|\dot{\gamma}(t)\|_{st} dt \leq k_1(k_2 + 1)(a_1\mu(I) + a_2)$$

Proof. First assume that either $I \cap \bar{I} = \emptyset$ or the conjugation on I is longitudinal as in Definition 2.5. Let $e : [0, \text{Mod} I] \rightarrow \mathbb{R}$ be given by

$$e(t) = \mu(S(0, t; I)).$$

Let $M = \lfloor \frac{2\mu(I)}{\delta_2} \rfloor$, let $0 = \alpha_0 < \alpha_1 < \dots < \alpha_M$ be the sequence defined by $e(\alpha_i) = i\frac{\delta_2}{2}$ and let $\alpha_{M+1} = L$. By our current assumption, $S(\alpha_i, \alpha_{i+1})$ is clean. Therefore, by Lemma 3.5 there is a constant $K \geq 1$ such that for any $1 \leq j \in \mathbb{N}$, any $0 \leq i \leq M$ such that $\alpha_{i+1} - \alpha_i > 2jK$ and any $x \in S(\alpha_i + jK, \alpha_{i+1} - jK; I)$, we have

$$f(x) \leq 2^{-j}.$$

Therefore, we calculate

(43)

$$\begin{aligned} \int_0^1 f(\gamma(t)) \|\dot{\gamma}(t)\|_{st} dt &= \sum_{i=0}^M \int_{\gamma^{-1}((\alpha_i, \alpha_{i+1}) \times S^1)} f(\gamma(t)) \|\dot{\gamma}(t)\|_{st} dt \\ &\leq \sum_{i=0}^M \left(4\pi k_2 k_1 K + \sum_{j=1}^{\lfloor \frac{\alpha_{i+1} - \alpha_i}{L} \rfloor} 4\pi k_1 K \times 2^{-j} \right) \\ &\leq \sum_{i=0}^M 4\pi k_1 (k_2 + 1) K \end{aligned}$$

By Lemma 2.6 it remains only to treat the case where the conjugation on I is latitudinal. Denote $I_1 = S(0, \frac{1}{2}\text{Mod} I; I)$ and $I_2 = S(\frac{1}{2}\text{Mod} I, \text{Mod} I; I)$. Then $I_i \cap \bar{I}_i = \emptyset$ for $i = 1, 2$. So, by what has

already been proved, the claim of the lemma holds for I_1 and I_2 separately. But up to addition of a constant the claim is additive. Thus the claim holds for $I = I_1 \cup I_2$. \square

For the following few lemmas let $\Sigma_{\mathbb{C}}$ be a surface with $\text{genus}(\Sigma_{\mathbb{C}}) > 1$ equipped with the metric h_{can} of constant curvature -1 , and let $\beta \subset \Sigma_{\mathbb{C}}$ be a simple closed geodesic. Recall the notation of Theorem 2.7. Let $I = \mathcal{C}(\beta)$. Denote by (ρ, θ) the coordinates on I given by Theorem 2.7(d). Note that ρ gives the distance from β . The coordinates (ρ, θ) are axially symmetric in the sense of equation (9) with

$$(44) \quad h_{\theta}(\rho) = \frac{\ell(\beta) \cosh \rho}{2\pi}.$$

Lemma 7.3. *There is a constant c with the following significance. For any $x \in I$,*

$$\frac{1}{\pi} \leq \frac{h_{\theta}(x)}{\text{InjRad}(\Sigma_{\mathbb{C}}; h_{can}, x)} \leq c.$$

Proof. First we prove the lower bound. Indeed, the non-contractible loop $\rho = \rho(x)$ has h_{can} -length $2\pi h_{\theta}(x)$. So, there is a non-constant geodesic beginning and ending at x with length less than $2\pi h_{\theta}(x)$. Therefore, $\text{Injrad}(\Sigma_{\mathbb{C}}; h_{can}, x) \leq \pi h_{\theta}(x)$ as desired.

We turn now to the proof of the upper bound. Denote $b = \frac{1}{2}\ell(\beta)$. Let $d_0(b) = w(\beta)$ and for any $x \in I$, let $d(x) = w(\beta) - \rho(x)$. By Theorems 2.7 and 2.8, we need only bound the expression

$$E(d, b) = \frac{b \cosh \rho}{\pi \sinh^{-1}(\cosh b \cosh d - \sinh d)}$$

in the region $0 \leq \rho \leq d_0(b)$. We have

$$(45) \quad \begin{aligned} E(d, b) &\leq \frac{b \cosh \rho}{\pi \sinh^{-1}(e^{-d})} \\ &\leq \frac{be^{\rho}}{2\pi \sinh^{-1}(e^{-d})} \\ &= \frac{be^{\rho}}{2\pi(e^{-d} + o(e^{-d}))} \\ &= \frac{be^{\rho+d}}{2\pi(1 + o(1))} \\ (46) \quad &= \frac{be^{d_0(b)}}{2\pi(1 + o(1))}. \end{aligned}$$

Pick a d' large enough that

$$|\sinh^{-1}(e^{-d'}) - e^{-d'}| \leq \frac{1}{2}.$$

A bound

$$(47) \quad \frac{be^{d_0(b)}}{2\pi} \leq c',$$

combined with bound (46) suffices to give the desired bound on $E(d, b)$ in the region $d > d'$. On the other hand, for $d \leq d'$ the desired bound on $E(d, b)$ follows from inequalities (45) and (47) since $\rho \leq d_0(b)$. To prove inequality (47), we use Theorem 2.7(c) to calculate

$$\frac{be^{d_0(b)}}{2\pi} = \frac{b}{2\pi} \left(\frac{1}{\sinh b} + \sqrt{\frac{1}{\sinh^2 b} + 1} \right),$$

which is clearly bounded for $b \in [0, \infty)$. □

Lemma 7.4. *Let $\epsilon \leq \pi$.*

- (a) $B_{\epsilon/2}(x; h_{st}) \subset B_{\epsilon h_\theta(x)}(x; h_{can})$.
- (b) *Let $\gamma \subset I$ be a embedded 1-manifold possibly with boundary. If $x_1, x_2 \in \gamma$ satisfy $d_\gamma(x_1, x_2; h_{st}) \geq \epsilon/2$, then*

$$d_\gamma(x_1, x_2; h_{can}) \geq \epsilon h_\theta(x_1).$$

Proof. (a) Let τ be the function on I defined by

$$\tau(y) = \tau(\rho(y)) = \int_{\rho(x)}^{\rho(y)} \frac{d\rho}{h_\theta(\rho)}.$$

So, $\tau \times \theta$ is a conformal diffeomorphism of I to the flat cylinder. In particular,

$$h_{st} = d\theta^2 + d\tau^2.$$

Suppose $y \in B_{\epsilon/2}(x; h_{st})$. First, we show that

$$(48) \quad |\rho(y) - \rho(x)| \leq \epsilon h_\theta(x).$$

Indeed, assume the contrary. By the formula (44) for h_θ , and the fact from Theorem 2.7(c) that

$$|\rho| \leq w(\beta) = \sinh^{-1}(1/\sinh(\ell(\beta)/2)),$$

we have

$$(49) \quad \left| \frac{dh_\theta}{d\rho}(\rho) \right| = \left| \frac{\ell(\beta) \sinh \rho}{2\pi} \right| \leq \frac{1}{\pi}.$$

So,

$$\begin{aligned}
|\tau(y)| &= \left| \int_{\rho(x)}^{\rho(y)} \frac{d\rho}{h_\theta(\rho)} \right| \\
&\geq \left| \int_{\rho(x)}^{\rho(y)} \frac{d\rho}{h_\theta(x) + \frac{1}{\pi}|\rho - \rho(x)|} \right| \\
&> \int_{|\rho(x)|}^{|\rho(x)| + \epsilon h_\theta(x)} \frac{d\rho}{h_\theta(x) + \frac{\epsilon h_\theta(x)}{\pi}} \\
&\geq \frac{\epsilon}{2},
\end{aligned}$$

which is a contradiction. Inequality (48) follows. Combining inequalities (48) and (49), we obtain

$$h_\theta(y) \leq 2h_\theta(x), \quad \forall y \in B_{\epsilon/2}(x; h_{st}).$$

So, since $h_{can} = h_\theta^2 h_{st}$, on $B_{\epsilon/2}(x; h_{st})$ we have the inequality of bilinear forms $h_{can} \leq (2h_\theta(x))^2 h_{st}$. Therefore, for $y \in B_{\epsilon/2}(x; h_{st})$, and α an h_{st} geodesic from x to y , we have

$$\ell_{h_{can}}(\alpha) \leq 2h_\theta(x)\ell_{h_{st}}(\alpha) \leq h_\theta(x)\epsilon.$$

That is, $y \in B_{\epsilon h_\theta(x)}(x; h_{can})$.

- (b) By way of contradiction, suppose $d_\gamma(x_1, x_2; h_{can}) < \epsilon h_\theta(x_1)$. Then the segment a in γ between x_1 and x_2 must be contained in $B_{\epsilon h_\theta(x_1)}(x_1; h_{can})$. Moreover, by inequality (49) we have

$$h_\theta(y) \leq 2h_\theta(x_1), \quad \forall y \in B_{\epsilon h_\theta(x_1)}(x_1; h_{can}).$$

So $h_{st} \leq (2h_\theta(x_1))^{-2} h_{can}$ on $B_{\epsilon h_\theta(x_1)}(x_1; h_{can})$. It follows that

$$\ell_{h_{st}}(a) \leq (2h_\theta(x_1))^{-1} \ell_{h_{can}}(a) < \frac{\epsilon}{2},$$

which is a contradiction. □

Lemma 7.5. *There is a constant C with the following significance. Let $k \geq 1$ and let γ be a compact embedded geodesic in I such that*

$$SegWidth(\gamma, x; h_{can}) \geq \frac{1}{k} InjRad(\Sigma_{\mathbb{C}}; h_{can}, x).$$

for all $x \in \gamma$. Then γ is Ck -tame.

Proof. Let $I' \subset I$ with $Mod(I') \geq 1$. Choose a collection of points $x_1, \dots, x_n \in \gamma \cap I'$ maximal with respect to the condition that

$$d_\gamma(x_i, x_j; h_{st}) \geq \frac{\pi}{2k}, \quad i \neq j.$$

In particular,

$$(50) \quad \ell_{h_{st}}(\gamma \cap I') < \frac{n\pi}{k}.$$

By Lemmas 7.4(b) and 7.3, we have

$$\begin{aligned} d_\gamma(x_i, x_j; h_{can}) &\geq \frac{\pi}{k} \max(h_\theta(x_i), h_\theta(x_j)) \\ &\geq \frac{1}{k} \max(InvRad(x_i), InvRad(x_j)). \end{aligned}$$

So, by the assumption on *SegWidth*,

$$B_{\frac{InvRad(x_i)}{2k}}(x_i; h_{can}) \cap B_{\frac{InvRad(x_j)}{2k}}(x_j; h_{can}) = \emptyset, \quad i \neq j.$$

Thus by Lemma 7.3 we have

$$B_{\frac{h_\theta(x_i)}{2ck}}(x_i; h_{can}) \cap B_{\frac{h_\theta(x_j)}{2ck}}(x_j; h_{can}) = \emptyset, \quad i \neq j.$$

It follows by Lemma 7.4(a) that

$$(51) \quad B_{\frac{1}{4ck}}(x_i; h_{st}) \cap B_{\frac{1}{4ck}}(x_j; h_{st}) = \emptyset, \quad i \neq j.$$

Consider the case $Mod(I') \geq 1$. Even if $x_i \in \partial I'$, at least half of $B_{\frac{1}{4ck}}(x_i; h_{st})$ is contained in I' . So,

$$Area\left(B_{\frac{1}{4ck}}(x_i; h_{st}) \cap I'\right) \geq \frac{\pi}{32c^2k^2}, \quad Area(I') = 2\pi Mod(I').$$

Therefore, by equation (51) we have

$$(52) \quad n \leq 64c^2k^2 Mod(I').$$

Combining inequalities (50) and (52), we deduce the lemma with

$$C = 32c^2.$$

On the other hand, suppose $Mod(I') < 1$. Then

$$Area\left(B_{\frac{1}{4ck}}(x_i; h_{st}) \cap I'\right) \geq \frac{\pi}{32c^2k^2} Mod I'.$$

So, by equation (51) we have

$$(53) \quad n \leq 64c^2k^2.$$

Combining inequalities (50) and (53), the lemma follows with the same value of C . \square

8. PROOF OF THE THEOREM

The goal of this section is to prove Theorem 4.2. We use the notation and terminology of Theorem 4.2 as well as Sections 5, 6, and 7. To begin, observe that

$$(54) \quad \begin{aligned} \ell_\mu(\gamma) &= \int_{x \in \gamma} e^{g(x)} d\ell_n(x) \\ &= \int_{x \in \gamma} \min\{e^{g(x)}, 2k\} d\ell_n(x) + \int_{x \in \gamma} \max\{e^{g(x)} - 2k, 0\}. \end{aligned}$$

We denote the first term by I_0 and the second by I_1 and deal with each term separately.

We first dispense with I_0 .

Lemma 8.1. *There are constants d_1 and d_2 such that*

$$I_0 \leq k^2 \{d_1 \mu(\Sigma_{\mathbb{C}}) + d_2 \text{genus}(\Sigma_{\mathbb{C}})\}.$$

Proof. Let

$$\text{Thick}(\Sigma_{\mathbb{C}}) := \{x \in \Sigma_{\mathbb{C}} \mid \text{InjRad}(x; h_{\text{can}}) \geq \sinh^{-1}(1)\},$$

and

$$\text{Thin}(\Sigma_{\mathbb{C}}) = \Sigma_{\mathbb{C}} \setminus \text{Thick}(\Sigma_{\mathbb{C}}).$$

In the following write $\Gamma = \text{genus}(\Sigma_{\mathbb{C}})$. Note that when $\Gamma > 1$, Theorem 2.8 implies $\text{Thin}(\Sigma_{\mathbb{C}})$ is contained in the union of at most $3\Gamma - 3$ clean geodesic cylinders. Cleanness follows from the conjugation invariance of h_{can} . When $\Gamma = 1$, $\text{Thin}(\Sigma_{\mathbb{C}})$ is either empty or all of $\Sigma_{\mathbb{C}}$. So, $\text{Thin}(\Sigma_{\mathbb{C}})$ is contained in the union of at most 2 clean geodesic cylinders. When $\Gamma = 0$, $\text{Thin}(\Sigma_{\mathbb{C}})$ is always empty. So, we define

$$N_\Gamma = \begin{cases} 3\Gamma - 3, & \Gamma \geq 2, \\ 2, & \Gamma = 1, \\ 0, & \Gamma = 0. \end{cases}$$

We calculate

$$(55) \quad \begin{aligned} I_0 &= \int_{x \in \gamma} \min\{e^{g(x)}, 2k\} d\ell_n(x) \\ &\leq \int_{\text{Thick}(\Sigma_{\mathbb{C}}) \cap \gamma} \frac{2k}{\sinh^{-1}(1)} d\ell_{h_{\text{can}}} + \int_{\text{Thin}(\Sigma_{\mathbb{C}}) \cap \gamma} \min\{e^{g(x)}, 2k\} d\ell_n(x) \end{aligned}$$

Let $\chi(\Sigma_{\mathbb{C}})$ denote the Euler characteristic and let

$$M_\Gamma = \max(|\chi(\Sigma_{\mathbb{C}})|, 1).$$

We claim that

$$(56) \quad \int_{Thick(\Sigma_{\mathbb{C}}) \cap \gamma} \frac{2k}{\sinh^{-1}(1)} d\ell_{h_{can}} \leq \frac{2k^2 M_{\Gamma}}{\pi(\sinh^{-1}(1))^2}.$$

Indeed, let $x_1, \dots, x_n \in \gamma \cap Thick(\Sigma_{\mathbb{C}})$ be a collection of points maximal with respect to the condition that

$$d_{\gamma}(x_i, x_j; h_{can}) \geq \frac{\sinh^{-1}(1)}{k}, \quad i \neq j.$$

In particular,

$$(57) \quad \ell_{h_{can}}(\gamma \cap Thick(\Sigma_{\mathbb{C}})) < 2n \frac{\sinh^{-1}(1)}{k}.$$

By assumption (20), we have

$$B_{\frac{\sinh^{-1}(1)}{k}}(x_i; h_{can}) \cap B_{\frac{\sinh^{-1}(1)}{k}}(x_j; h_{can}) = \emptyset, \quad i \neq j.$$

Moreover,

$$Area\left(B_{\frac{\sinh^{-1}(1)}{k}}(x_i; h_{can})\right) \geq \pi \left(\frac{\sinh^{-1}(1)}{k}\right)^2, \quad Area(\Sigma_{\mathbb{C}}) = M_{\Gamma}.$$

So,

$$(58) \quad n \leq \frac{k^2 M_{\Gamma}}{\pi(\sinh^{-1}(1))^2}.$$

Combining inequalities (57) and (58) we obtain

$$\ell_{h_{can}}(\gamma \cap Thick(\Sigma_{\mathbb{C}})) < \frac{k M_{\Gamma}}{\pi \sinh^{-1}(1)},$$

which implies inequality (56) as claimed.

Furthermore, we calculate

$$(59) \quad \begin{aligned} \int_{Thin(\Sigma_{\mathbb{C}}) \cap \gamma} \min\{e^{g(x)}, 2k\} d\ell_n(x) &\leq \\ &\leq \int_{Thin(\Sigma_{\mathbb{C}}) \cap \gamma} \min\{e^{g(x)}, 2k\} \frac{d\ell_{h_{can}}(x)}{InjRad(x)} \\ &= \int_{Thin(\Sigma_{\mathbb{C}}) \cap \gamma} \min\{e^{g(x)}, 2k\} \frac{h_{\theta}(x)}{InjRad(x)} d\ell_{h_{st}}(x) \\ &\leq Ck(2k+1)(a_2 N_{\Gamma} a_1 \mu(\Sigma_{\mathbb{C}})) \end{aligned}$$

using Lemmas 7.2, 7.5 and 7.3 in the last inequality. Combining inequalities (55), (56), and (59), we obtain

$$(60) \quad I_0 \leq \frac{2k^2 M_{\Gamma}}{\pi(\sinh^{-1}(1))^2} + Ck(2k+1)(a_2 N_{\Gamma} + a_1 \mu(\Sigma_{\mathbb{C}})).$$

Note that when $\Gamma = 0$, the first term in the right hand side of (60) can be absorbed in the coefficient d_1 . This is so because the gradient inequality implies that for any $(\Sigma, \mu) \in \mathcal{M}$ with $\text{genus}(\Sigma_{\mathbb{C}}) = 0$, we have $\mu(\Sigma) \geq \delta_1$. \square

Assumption 8.2. *From here until Lemma 8.10, we assume that conditions (30) and (31) are satisfied. Thus we may use the constructions and results of Section 6.*

The following arguments deal with I_1 given the preceding assumption.

$$\begin{aligned} I_1 &= \int_{x \in \gamma} \max\{e^{g(x)} - 2k, 0\} d\ell_n(x) \\ &= \int_{x \in \gamma} \int_{2k}^{\max\{e^{g(x)}, 2k\}} dy d\ell_n(x). \end{aligned}$$

Change variables to $y = e^t$. Then, using Fubini's theorem,

$$\begin{aligned} (61) \quad I_1 &= \int_{\ln 2k}^{\infty} \int_{x \in D_t} d\ell_n(x) e^t dt \\ &= \int_{\ln 2k}^{\infty} \int_{x \in D_t \cap C} d\ell_n(x) e^t dt + \int_{\ln 2k}^{\infty} \int_{x \in D_t \cap A} d\ell_n(x) e^t dt \\ &\leq \int_{\ln 2k}^{\infty} \int_{x \in C_t} d\ell_n(x) e^t dt + \int_{\ln 2k}^{\infty} \int_{x \in D_t \cap A} d\ell_n(x) e^t dt. \end{aligned}$$

In the last line, denote the first term, giving the contribution of the thick part, by I_2 , and the second, giving the contribution of the thin part, by I_3 .

Lemma 8.3. *There is a constant d_3 such that $I_2 \leq d_3 \mu(\Sigma_{\mathbb{C}})$.*

Proof. Discretizing, we have the bound

$$(62) \quad I_2 \leq (2 \ln 3)(2k) \sum_{i=0}^{\infty} 3^{2(i+1)} \ell_n\{C_{2i \ln 3 + \ln 2k}\}.$$

Write $c_i := C_{2i \ln 3 + \ln 2k}$ and denote by c_{ij} the components of c_i . By Lemma 6.27, for each j choose

$$t_{ij} \in [2i \ln 3 + \ln 2k, (2i + 1) \ln 3 + \ln 2k)$$

and a component $k_{ij} \subset K_{t_{ij}}$ such that $c_{ij} \leq k_{ij}$. In the case that $c_{ij} \subset K$, we insist upon taking $t_{ij} = 2i \ln 3 + \ln 2k$ and $k_{ij} = c_{ij}$. For each j let $T_{ij} \subset k_{ij}$ be a segment of length

$$(63) \quad \ell_n(T_{ij}) = \frac{8}{e^{t_{ij}}} \left\lfloor \frac{e^{t_{ij}} \ell_n(k_{ij})}{8} \right\rfloor.$$

Since $k_{ij} \subset K_{t_{ij}}$, we have

$$(64) \quad \ell_n(k_{ij}) \geq 24e^{-t_{ij}}.$$

So,

$$(65) \quad \ell_n(T_{ij}) \geq \frac{3}{4}\ell_n(k_{ij}).$$

Partition T_{ij} into segments T_{ijk} such that

$$(66) \quad \ell_n(T_{ijk}) = 8e^{-t_{ij}}.$$

Let n_{ij} be the number of segments T_{ijk} . We claim that

$$(67) \quad \frac{8n_{ij}}{3^{2i}2k} \geq \frac{3}{4}\ell_n(c_{ij}).$$

Indeed, suppose first that $c_{ij} = k_{ij}$. Then $t_{ij} = 3^{2i}2k$. So, using inequality (65), we have

$$\frac{8n_{ij}}{3^{2i}2k} = 8e^{-t_{ij}}n_{ij} = \ell_n(T_{ij}) \geq \frac{3}{4}\ell_n(k_{ij}) = \frac{3}{4}\ell_n(c_{ij}).$$

On the other hand, suppose $c_{ij} \neq k_{ij}$. Then $c_{ij} \subset N$, so $\ell(c_{ij}) \leq \frac{24}{3^{2i}2k}$. But

$$n_{ij} = \frac{e^{t_{ij}}}{8}\ell_n(T_{ij}) \geq \frac{3}{4}\frac{e^{t_{ij}}}{8}\ell_n(k_{ij}) \geq \frac{3}{4}\frac{24}{8} > 2.$$

Since n_{ij} is an integer, we have $n_{ij} \geq 3$. Thus

$$\frac{8n_{ij}}{3^{2i}2k} \geq \frac{24}{3^{2i}2k} \geq \ell(c_{ij}).$$

Inequality (67) follows. Setting

$$(68) \quad n_i = \sum n_{ij},$$

inequality (67) implies

$$(69) \quad \frac{8n_i}{3^{2i}2k} \geq \frac{3}{4}\ell_n(C_{2i \ln 3 + \ln 2k}).$$

To each segment T_{ijk} , assign a disk $B_{ijk} \subset \Sigma_{\mathbb{C}}$ as in Remark 6.1. So, if $k_1 \neq k_2$, then $B_{ijk_1} \cap B_{ijk_2} = \emptyset$. Let m_i denote the number of disks $B_{i'j'k'}$ with $i' < i$ that intersect one of the disks B_{ijk} but do not intersect any disk $B_{i''j''k''}$ for $i' < i'' < i$. We claim that

$$(70) \quad m_i \leq \frac{3}{4}n_i.$$

Indeed, let m'_i denote the number of intervals $T_{i'j'k'}$ with $i' < i$ such that $p_1(T_{i'j'k'}) \cap p_1(T_{ijk}) \neq \emptyset$ for some j, k , and $p_1(T_{i'j'k'}) \cap p_1(T_{i''j''k''}) = \emptyset$

for $i' < i'' < i$. By Remark 6.1 we have $m_i \leq m'_i$. So inequality (70) will follow if we show

$$(71) \quad m'_i \leq \frac{3}{4}n_i.$$

Let m'_{ij} denote the number of intervals $T_{i'j'k'}$ with $i' < i$ such that $p_1(T_{i'j'k'}) \cap p_1(T_{ijk}) \neq \emptyset$ for some k and

$$(72) \quad p_1(T_{i'j'k'}) \cap p_1(T_{i''j''k''}) = \emptyset$$

for $i' < i'' < i$. Then

$$m'_i \leq \sum_j m'_{ij}.$$

So, by equation (68) inequalities (71) and (70) will follow if we prove

$$(73) \quad m'_{ij} \leq \frac{3}{4}n_{ij}$$

for all j . To see this, choose integers a_1 and $0 \leq a_2 < 3$ such that $n_{ij} = 3a_1 + a_2$. By equations (63), (64) and (66), we have $a_1 \geq 1$. Enumerate the intervals $T_{i'j'k'}$ by $T_{i_l j_l k_l}$ for $l = 1, \dots, m'_{ij}$. By (72) we conclude the intervals $p_1(T_{i_l j_l k_l})$ are pairwise disjoint. Since $i_l < i$ for all l , by equation (66) we have $\ell_n(T_{i_l j_l k_l}) \geq 3\ell_n(T_{ijk})$. Therefore it is easy to verify that m'_{ij} is at most $a_1 + 2$ when $a_2 \neq 0$, and at most $a_1 + 1$ when $a_2 = 0$. But in the first case,

$$\frac{m'_{ij}}{n_{ij}} = \frac{a_1 + 2}{3a_1 + a_2} \leq \frac{3}{4},$$

and in the second case,

$$\frac{m'_{ij}}{n_{ij}} = \frac{a_1 + 1}{3a_1} \leq \frac{2}{3} \leq \frac{3}{4}.$$

Inequalities (73), (71) and (70) follow.

Combining inequalities (70), (69) and (62), we obtain

$$(74) \quad \begin{aligned} \frac{\mu(\Sigma_{\mathbb{C}})}{\delta_1} &\geq \sum_{i=0}^{\infty} n_i - m_i \\ &\geq \sum_{i=0}^{\infty} \frac{1}{4}n_i \\ &\geq \frac{2k}{4 \cdot 8 \cdot 4} \sum_{i=0}^{\infty} 3^{2i+1} \ell_n(C_{2i \ln 3 + \ln 2k}) \\ &\geq \frac{1}{256 \ln 3} I_2. \end{aligned}$$

□

We now deal with the thin part.

Lemma 8.4. *There is a constant d_4 such that $I_3 \leq d_4\mu(\Sigma_{\mathbb{C}})$.*

Proof. Recall that G denotes the collection of all non-exceptional thin necks. Let $G_1 = \{v \in G \mid T(v) \leq \ln 48\}$. The contribution of the thin part is

$$\begin{aligned}
I_3 &= \int_{\ln 2k}^{\infty} \int_{x \in A_t \cap D} d\ell_n(x) e^t dt \\
&= \sum_{v \in G} \int_{T_i(v)}^{T_f(v)} \int_{x \in v_t \cap D} d\ell_n(x) e^t dt \\
&\leq \sum_{v \in G_1} \int_{T_i(v)}^{T_f(v)} 24 dt + \sum_{v \in G \setminus G_1} \int_{T_i(v)}^{T_f(v)} \int_{x \in v_t \cap D} d\ell_n(x) e^t dt \\
&\leq 24 \ln 48 \frac{12\mu(\Sigma_{\mathbb{C}})}{\delta_1} + \sum_{v \in G \setminus G_1} \int_{T_i(v)}^{T_f(v)} \int_{x \in v_t \cap D} d\ell_n(x) e^t dt.
\end{aligned}$$

The last transition relies on Lemma 6.34.

To each $v \in G \setminus G_1$ we associate an annulus as follows. Let x_v denote the midpoint of $v_{T_f(v)}$ with respect to the metric h_{can} and let

$$r_v = \frac{1}{2} \ell_{h_{can}}(v_{T_f(v)}).$$

Let

$$\begin{aligned}
r_v^1 &= r_v + 2r(x_v)e^{-T_f(v)}, \\
r_v^2 &= r(x_v)e^{-T_i(v)},
\end{aligned}$$

$$B_v^1 := B_{r_v^1}(x_v; h_{can}) \subset \Sigma_{\mathbb{C}}, \quad B_v^2 = B_{r_v^2}(x_v; h_{can}) \subset \Sigma_{\mathbb{C}},$$

and $A_v := B_v^2 \setminus B_v^1$.

The following claims justify the definitions of the preceding paragraph.

Claim 8.5. $r_v^1 < r_v^2 \leq \frac{1}{2k}r(x_v) < \frac{1}{2}\text{SegWidth}(\gamma, x_v; h_{can})$.

Proof. The inequality $r_v^2 \leq \frac{1}{2k}r(x_v)$ follows from the fact that $T_i(v) \geq \ln 2k$. The inequality $\frac{1}{2k}r(x_v) < \frac{1}{2}\text{SegWidth}(\gamma, x_v; h_{can})$ is a special case of assumption (20). By inequality (22) and the definition of N ,

$$r_v \leq \frac{3}{4} \ell_n(v_{T_f(v)}) r(x_v) \leq \frac{3}{4} \cdot 24e^{-T_f(v)} r(x_v).$$

Therefore, $r_v^1 \leq 20r(x_v)e^{-T_f(v)}$. Since

$$T_f(v) - T_i(v) = T(v) > \ln 20,$$

the inequality $r_v^1 < r_v^2$ follows. \square

Claim 8.6. *Let $t \geq T_i(v) + \ln 48$. Then $p_1(v_t) \subset B_v^2$.*

Proof. This is immediate from inequality (22) and the definition of N . \square

Claim 8.7. *For any $v_1, v_2 \in G \setminus G_1$, we have*

$$v_1 \neq v_2 \Rightarrow A_{v_1} \cap A_{v_2} = \emptyset.$$

Proof. Consider first the case that $p_1(v_1) \cap p_1(v_2) \neq \emptyset$. By Lemmas 5.12 and 6.13(b), we may assume without loss of generality that

$$(v_1)_{T_f(v_1)} \leq v_2.$$

So, we have

$$(75) \quad d(x_{v_1}, x_{v_2}; h_{can}) \leq r_{v_1}.$$

Thus by Claim 8.5 we obtain

$$d(x_{v_1}, x_{v_2}; h_{can}) < r(x_{v_1}).$$

So, it follows from Lemma 4.5 that

$$r(x_{v_2}) \leq 2r(x_{v_1}).$$

Thus we have

$$r_{v_1}^2 = r(x_{v_2})e^{-T_i(v_2)} \leq 2r(x_{v_1})e^{-T_f(v_1)}.$$

The preceding inequality and inequality (75) together imply

$$d(x_{v_1}, x_{v_2}; h_{can}) + r_{v_1}^2 \leq r_{v_1} + 2r(x_{v_1})e^{-T_f(v_1)}.$$

It follows that $B_{v_2}^2 \subset B_{v_1}^1$ giving the claim.

On the other hand, if $p_1(v_1) \cap p_2(v_2) = \emptyset$, Corollary 6.9 implies

$$d(x_{v_1}, x_{v_2}; h_n) \geq 4e^{-\min\{T_i(v_1), T_i(v_2)\}}.$$

But by inequality (22) and the definition of $B_{v_j}^2$ for $j = 1, 2$, we have the inequality

$$d(x_{v_j}, \partial B_{v_j}^2 \cap \gamma; h_n) \leq 2e^{-T_i(v_j)}.$$

Therefore, $B_{v_1}^2 \cap B_{v_2}^2 \cap \gamma = \emptyset$. By definition of *SegWidth* and by Claim 8.5, we conclude $B_{v_1}^2 \cap B_{v_2}^2 = \emptyset$. \square

Claim 8.8. *A_v is clean.*

Proof. Keeping in mind Remark 6.35, by construction $A_{\bar{v}} = \overline{A_v}$. In particular if $v = \bar{v}$, then A_v is conjugation invariant. If $v \neq \bar{v}$, then by Claim 8.7 we have $A_v \cap \overline{A_v} = \emptyset$. \square

Claim 8.9.

$$(76) \quad \ell_n(B_v^1 \cap \gamma) \leq 32e^{-T_f(v)}.$$

Proof. We have

$$\ell_n(B_v^1 \cap \gamma) = \ell_n(v_{T_f(v)}) + \ell_n(B_v^1 \cap \gamma \setminus p_1(v_{T_f(v)})).$$

Denote the first term by a and the second term by b . The definition of N implies that $a \leq 24e^{-T_f(v)}$. Since $T(v) \geq \ln 48$, we have $2r(x_v)e^{-T_f(x_v)} \leq \frac{1}{2}r(x_v)$. So, we use inequality (22) to verify that $b \leq 8e^{-T_f(v)}$. \square

We return to estimating I_3 . By the inclusion of Claim 8.6, for $v \in G \setminus G_1$ we obtain

$$\begin{aligned} \int_{T_i(v)}^{T_f(v)} \int_{x \in p_1(v_t \cap D)} d\ell_n(x) e^t dt &\leq \\ &\leq 24 \ln 48 + \int_{T_i(v) + \ln 48}^{T_f(v)} \int_{x \in p_1(v_t \cap D)} d\ell_n(x) e^t dt \\ &\leq 24 \ln 48 + \int_0^{T_f(v)} \int_{x \in B_v^2 \cap p_1(v_t \cap D)} d\ell_n(x) e^t dt. \end{aligned}$$

By equation (76) we have

$$\begin{aligned} \int_0^{T_f(v)} \int_{x \in B_v^2 \cap p_1(v_t \cap D)} d\ell_n(x) e^t dt &\leq \\ &\leq \int_{x \in A_v \cap \gamma} e^{g(x)} d\ell_n(x) + \int_{x \in B_v^1} e^{T_f(v)} d\ell_n(x) \\ &\leq \int_{x \in A_v \cap \gamma} e^{g(x)} d\ell_n(x) + 32. \end{aligned}$$

We bound the integral on the domain $A_v \cap \gamma$ as follows. By Lemma 6.13(d), we have

$$\frac{d\ell_\mu}{d\ell_n}(x) = e^{g(x)} \leq \frac{24}{d(x, x_v; h_n)},$$

for any $x \in A_v \cap \gamma$. So, by inequality (22) we deduce that

$$(77) \quad r(x_v) \frac{d\ell_\mu}{d\ell_{h_{can}}}(x) \leq \frac{36r(x_v)}{d(x, x_v; h_{can})}.$$

In polar coordinates on B_v^2 , the metric h_{can} is given by formulas (10) and (11), and $h_{can}|_{A_v} = h_\theta^2 h_{st}$. So, since by Claim 8.5 we have $r_v^2 \leq 1$,

it follows that

$$(78) \quad \frac{d\ell_{h_{can}}}{d\ell_{h_{st}}}(x) = h_\theta(x) \leq 3d(x, x_v; h_{can}), \quad x \in A_v.$$

Using the chain rule to combine inequalities (77) and (78), we obtain

$$\frac{d\ell_\mu}{d\ell_{h_{st}}}(x) \leq 108$$

for $x \in A_v \cap \gamma$. On the other hand, we have

$$\int_{x \in A_v} e^{g(x)} d\ell_n(x) = \int_{x \in A_v} \frac{d\ell_\mu}{d\ell_{h_{st}}} d\ell_{h_{st}}.$$

It follows from Claim 8.5 that $\gamma \cap B_v^2$ is a radial geodesic, so $\gamma \cap A_v$ is 1-tame. Thus by Claim 8.8 we may apply Lemma 7.2 with $k_1 = 1$ and $k_2 = 108$ to deduce the bound

$$\int_{x \in A_v} e^{g(x)} d\ell_n(x) \leq 108(a_1\mu(A_v) + a_2).$$

Collecting the terms, using Claim 8.7, and invoking Lemma 6.34 for the constant terms, we obtain

$$(79) \quad \sum_{v \in G \setminus G_1} \int_{t_0(v)}^{t_1(v)} \int_{x \in v_t \cap D} d\ell_n(x) e^t dt \leq \left(\frac{12}{\delta_1} (24 \ln 48 + 108a_2 + 32) + 108a_1 \right) \mu(\Sigma_{\mathbb{C}}),$$

which completes the proof. \square

Lemma 8.10. *Let $\beta \subset \gamma$ be the union of the connected components β_i of γ such that*

$$(80) \quad \ell_n(\beta_i) < 4e^{-\max_{x \in \beta_i} g(x)},$$

and

$$(81) \quad \ln 2k \leq \max_{x \in \beta_i} g(x).$$

We have

$$\ell_\mu(\beta) \leq 4 \frac{\mu(\Sigma_{\mathbb{C}})}{\delta_1}.$$

Proof. Using inequality (80) we estimate

$$\ell_\mu(\beta_i) \leq \ell_n(\beta_i) \max_{x \in \beta_i} \frac{d\ell_\mu}{d\ell_n}(x) \leq 4.$$

Choose $x_i \in \beta_i$ such that

$$\frac{d\ell_\mu}{d\ell_n}(x_i) = \max_{x \in \beta_i} \frac{d\ell_\mu}{d\ell_n}(x).$$

Set $t_i = \frac{d\ell_\mu}{d\ell_n}(x_i)$. By inequality (81), we have $(x_i, t_i) \in D$. So, Lemma 4.7 implies that $\mu(B(x_i, t_i)) \geq \delta_1$, and Lemma 4.8 implies the disks $B(x_i, t_i)$ are pairwise disjoint implying the claim. \square

Proof of Theorem 4.2. By equation (54), it suffices to bound I_0 and I_1 . Lemma 8.1 takes care of I_0 unconditionally. The components $\gamma_i \subset \gamma$ that violate condition (31) contribute trivially to I_1 , so we assume without loss of generality that condition (31) does hold. Then Lemma 8.10 allows us to disregard the contribution to I_1 of components $\gamma_i \subset \gamma$ that violate condition (30). So, without loss of generality, we impose Assumption 8.2. Then equation (61), Lemma 8.3 and Lemma 8.4 bound I_1 implying Theorem 4.2. \square

9. APPLICATIONS

We now apply Theorem 4.2 to prove the theorems stated in the introduction.

Lemma 9.1. *Let Σ be a closed Riemann surface with an isometric involution ψ . Suppose $p \in \Sigma$ is fixed under ψ . Then for any $r > 0$, $B_r(p; h_{can})$ is ψ invariant.*

Proof. For any $q \in \Sigma$ we have

$$d(p, q; h_{can}) = d(\psi(p), \psi(q); h_{can}) = d(p, \psi(q); h_{can}).$$

In particular, $q \in B_r(p; h_{can})$ if and only if $\psi(q) \in B_r(p; h_{can})$. \square

Lemma 9.2. *Let Σ be a closed Riemann surface with an isometric involution ψ . Let γ be either a minimal geodesic connecting two points or an embedded geodesic fixed under an isometric involution ψ . Then for any $p \in \gamma$,*

$$SegWidth(\gamma, p; h_{can}) \geq InjRad(\Sigma; h_{can}, p).$$

Proof. If γ is a minimal geodesic the claim is obvious. Suppose γ is fixed under ψ . For any $p \in \gamma$ and $r \in (0, InjRad(\Sigma; h_{can}, p))$, write $B = B_r(p; h_{can})$. Let Σ_ψ denote the fixed points of ψ . B is conjugation invariant by Lemma 9.1. Denote by B_ψ the fixed points of $\psi|_B$. It is clear that B_ψ is a radial geodesic and that $B_\psi = \Sigma_\psi \cap B$. It follows that $B_r(p; h_{can}) \cap \gamma$ is contained in a radial geodesic and so, that $SegWidth(\gamma, p; h_{can}) \geq r$. \square

The following theorem is proved in [7, Section 4]. In cases (a) and (b), it follows straightforwardly from the discussion in Chapter 4 of [18].

Theorem 9.3. *Let (M, ω) be a symplectic manifold, $L \subset M$ a Lagrangian submanifold, and J an ω -tame almost complex structure. Let \mathcal{M} be a family of J -holomorphic curves in M with boundary in L . Suppose that one of the following conditions holds:*

- (a) *M and L are compact.*
- (b) *$L = \emptyset$ and there is a $K > 1$ such that the curvature of g_J and the derivatives of J are bounded from above by K , and the radius of injectivity is bounded from below by $\frac{1}{K}$.*
- (c) *There is a $K > 1$ such that the quadruple (M, ω, J, L) has K bounded geometry and L has a tubular neighborhood of width $\frac{1}{K}$.*
- (d) *There is a $K > 1$ such that the quadruple (M, ω, J, L) has K bounded geometry, and each connected component of L has a tubular neighborhood of width $\frac{1}{K}$. Furthermore, for each $(u, \Sigma) \in \mathcal{M}$, fix a conformal metric of constant curvature, of volume $1 + \text{genus}(\Sigma)$, and such that $\partial\Sigma$ is totally geodesic. Then, with respect to this metric, $\partial\Sigma$ has a tubular neighborhood of width $\frac{1}{K}$.*

For each $(u, \Sigma) \in \mathcal{M}$ let μ_u be the corresponding energy measure

$$\mu_u(U) := \int_U \|du\|^2 d\text{vol}_\Sigma,$$

for $U \subset \Sigma$ an open subset. Then the collection of measured Riemann surfaces

$$\{(\Sigma, \mu_u) | (u, \Sigma) \in \mathcal{M}\}$$

is uniformly thick thin.

Proof of Theorems 1.1, 1.2, 1.4 and 1.5. Theorems 1.1, 1.2, 1.4 and 1.5 are immediate from Theorem 4.2, Lemma 9.2, Remark 4.3 and Theorem 9.3. Note that Theorem 1.5 is covered by case (b) in Theorem 9.3. \square

To prove Theorem 1.6, we need the following additional definitions and lemmas.

Definition 9.4. Let Σ be a Riemann surface with boundary. A **bridge** in Σ is a length minimizing geodesic connecting two components $\gamma_1 \neq \gamma_2$ of $\partial\Sigma$. An **admissible bridge** is a bridge γ such that for any $p \in \gamma$ and any $r \in \frac{1}{3} \text{InjRad}(\Sigma_{\mathbb{C}}; h_{can}, p)$, we have $B_r(p; h_{can}) \cap \partial\Sigma \subset \gamma_1 \cup \gamma_2$.

In the following Lemmas 9.5 to 9.8, we shall consider a Riemann surface Σ with boundary and its complex double $\Sigma_{\mathbb{C}}$. All metric quantities

and sets are with respect to the metric h_{can} on $\Sigma_{\mathbb{C}}$, so we omit them from the notation.

Lemma 9.5. *Let γ be an admissible bridge in Σ connecting the components γ_1 and γ_2 of $\partial\Sigma$. Then for any $p \in \gamma \cup \bar{\gamma} \subset \Sigma_{\mathbb{C}}$,*

$$SegWidth(\gamma \cup \bar{\gamma}, p) \geq \frac{1}{3} InjRad(p).$$

Proof. Let $p \in \gamma$ and $r \in (0, \frac{1}{3} InjRad(p))$. Write

$$B = B_r(p),$$

and

$$B' := B_{3r}(p).$$

We need to show that $B \cap (\gamma \cup \bar{\gamma})$ is a radial geodesic. If $B \cap \partial\Sigma = \emptyset$ then $(\gamma \cup \bar{\gamma}) \cap B = \gamma \cap B$ is a minimizing geodesic, so the claim follows from Lemma 9.2. Otherwise, let q be the point of $\gamma \cap \partial\Sigma$ closest to p , and let

$$B'' := B_{2r}(q).$$

Since γ is admissible, $B \cap \partial\Sigma \subset \gamma_1 \cup \gamma_2$. Thus since γ minimizes length between γ_1 and γ_2 , we have $q \in B$. The triangle inequality implies that $B \subset B'' \subset B'$. Since h_{can} has constant curvature and B' is a normal disk, it follows that B'' is a normal disk. By Lemma 9.2 and the fact that B'' is a normal disk, we have that each of $\gamma \cap B''$ and $\bar{\gamma} \cap B''$ lies on a radial geodesic of B'' . Furthermore γ and $\bar{\gamma}$ intersect $\partial\Sigma$ at q in a right angle. So, $B'' \cap (\gamma \cup \bar{\gamma})$ is a radial geodesic which we denote by C . Since $p \in C$, again using the fact h_{can} has constant curvature, $B \cap C = B \cap (\gamma \cup \bar{\gamma})$ is a radial geodesic in B . \square

Definition 9.6. Let Σ be a Riemann surface with boundary and let γ_i , for $i = 1, 2$, be components of $\partial\Sigma$. An *admissible chain* connecting γ_1 and γ_2 is a pairwise disjoint sequence

$$\alpha_1 = \gamma_1, \alpha_2, \dots, \alpha_n = \gamma_2,$$

of components of $\partial\Sigma$ with admissible bridges β_i connecting α_i and α_{i+1} for each $1 \leq i \leq n-1$.

Lemma 9.7. *Let Σ be a Riemann surface with boundary and let χ_i , for $i = 1, 2$, be components of $\partial\Sigma$. There exists an admissible chain connecting χ_1 and χ_2 .*

Lemma 9.8. *Let β be a bridge connecting boundary components γ_1 and γ_2 . Suppose there is a boundary component γ_3 and a point $p \in \beta$ such that*

$$d(p, \gamma_3) < \frac{1}{3} InjRad(p).$$

Let β_i for $i = 1, 2$, be a bridge connecting γ_3 with γ_i . Then

$$\ell(\beta_i) < \ell(\beta).$$

Proof. Let δ be the length minimizing geodesic from p to γ_3 and let $q = \delta \cap \gamma_3$. Write $r = \text{InjRad}(p)$. Let $B_1 = B_r(p)$ and $B_2 = B_{\frac{2}{3}r}(q)$. By the triangle inequality, $B_2 \subset B_1$. Since h_{can} has constant curvature and B_1 is a normal disk, so is B_2 . In particular $\text{InjRad}(q) \geq \frac{2}{3}\text{InjRad}(p)$. It follows that

$$(82) \quad d(p, q) < \frac{1}{3}\text{InjRad}(p) \leq \frac{1}{2}\text{InjRad}(q).$$

On the other hand, by Lemma 9.2 applied to the Riemann surface $\Sigma_{\mathbb{C}}$ with conjugation as the isometric involution we have, for $i = 1, 2$,

$$(83) \quad d(q, \gamma_i) \geq \text{InjRad}(q).$$

Denote by p_i for $i = 1, 2$, the point where β meets γ_i . Combining inequalities (82), (83) and the triangle inequality,

$$(84) \quad d(p, p_i) \geq d(q, p_i) - d(p, q) \geq d(q, \gamma_i) - d(p, q) > \frac{1}{2}\text{InjRad}(q).$$

Therefore, since $d(p, p_1) + d(p, p_2) = \ell(\beta)$, we have

$$(85) \quad d(p, p_i) < \ell(\beta) - \frac{1}{2}\text{InjRad}(q).$$

Combining inequalities (82) and (85), we conclude

$$(86) \quad \ell(\beta_i) \leq d(p_i, q) \leq d(p_i, p) + d(p, q) < \ell(\beta).$$

□

Proof of Lemma 9.7. For any graph Γ , we denote by $\mathcal{E}(\Gamma)$ the set of edges of Γ . Let G be the complete graph with vertices the boundary components of Σ . For any $e \in \mathcal{E}(G)$, denote by $\ell(e)$ the length of a corresponding bridge. Let \mathcal{S} be the set of spanning trees of G . For $T \in \mathcal{S}$, we write

$$\ell(T) = \sum_{e \in \mathcal{E}(T)} \ell(e).$$

Let $T \in \mathcal{S}$ be a tree that minimizes $\ell(\cdot)$. We claim that all bridges corresponding to an edge of T are admissible. Indeed, suppose the contrary and let $e \in \mathcal{E}(T)$ correspond to a non-admissible bridge β connecting γ_1 and γ_2 . By definition, there is a γ_3 such that, using the notation of Lemma 9.8, the condition of Lemma 9.8 is satisfied. For $i = 1, 2$, let e_i be the edge of G connecting γ_3 and γ_i . Removing e disconnects T into two connected components T_1 and T_2 containing the vertices γ_1 and γ_2 respectively. Without loss of generality, suppose

the vertex corresponding to γ_3 is in T_2 . Then, connecting T_1 and T_2 with the edge e_1 produces a spanning tree T' . By Lemma 9.8 we have

$$\ell(T') = \ell(T) - \ell(e) + \ell(e_1) < \ell(T)$$

contrary to the choice of T . The claim follows. So, the sequence of boundary components corresponding to the unique path in T connecting χ_1 with χ_2 is an admissible chain. \square

Proof of Theorem 1.6. To deduce the diameter estimate of Theorem 1.6, let $p_1, p_2 \in \Sigma$. If $\partial\Sigma = \emptyset$, Theorem 4.2 and Lemma 9.2 immediately imply the claim. Otherwise, let δ_i , for $i = 1, 2$, be the minimal geodesic connecting p_i to $\overline{p_i}$. Clearly, δ_i is conjugation invariant and intersects a component γ_i of $\partial\Sigma$. Let $(\alpha_i)_{i \leq n}$ be an admissible chain connecting γ_1 and γ_2 with bridges β_i connecting α_i and α_{i+1} for $1 \leq i \leq n-1$. It follows from the definition of an admissible chain that $n \leq |\pi_0(\partial\Sigma)|$. We have the estimate

$$d(u(p_1), u(p_2)) \leq \ell(u|_{\delta_1}) + \ell(u|_{\delta_2}) + \ell(u|_{\partial\Sigma}) + \sum_{i=1}^{n-1} \ell(u|_{\beta_i}),$$

where all the lengths and distances are measured with respect to g_J . So, the claim again follows from Theorem 4.2, Lemmas 9.2 and 9.5, and Theorem 9.3. \square

Remark 9.9. The reader may have noted the unequal treatment of the elements α_i and β_i in the admissible chain connecting two boundary components. For the α_i we have an estimate which is independent of the number of components since Lemma 9.2 bounds the segment widths of the boundary as a whole. For the β_i , on the other hand, Lemma 9.5 only provides estimates for each component separately. For now we leave open the question whether or not this may be improved to eliminate the dependence on the number of boundary components in Theorem 1.6 altogether.

10. CALCULATION OF OPTIMAL ISOPERIMETRIC CONSTANT

Proof of Proposition 1.3. Let $M = \mathbb{CP}^n$, let $L = \mathbb{RP}^n$ and let $\omega = \omega_{FS}$ be the Fubini Study form, normalized so the area of line is $\frac{1}{\pi}$. Let J be an ω -tame almost complex structure. Let $\mathcal{M}_{0,2}(M, L, J)$ denote the moduli space of degree 1 J -holomorphic disk maps with two marked points on the boundary modulo reparametrization. Since the energy of a degree 1 map is minimal, there is no bubbling, and $\mathcal{M}_{0,2}(M, L, J)$ is compact. The structure theorem for the image of J -holomorphic disks [14, 15] implies that minimal energy disks are

somewhere injective. It follows that there is a dense set of regular J . Recall that for J regular, $\mathcal{M}_{0,2}(M, L, J)$ is a manifold. Let

$$ev_J : \mathcal{M}_{0,2}(M, L, J) \rightarrow L^2$$

be the evaluation map. We claim that ev_J is surjective for all J . To see this, note that for J regular ev_J is relatively orientable by [22, Theorem 1]. In particular, it is not hard to check that the standard complex structure J_{st} is regular. For $J = J_{st}$, degree 1 disk maps are equivalent to oriented real lines. So $ev_{J_{st}}$ is 2 to 1 away from the diagonal. Using [22, Prop. 5.1], one can check that conjugate disks contribute with equal sign, so $e_{J_{st}}$ has degree 2. A routine cobordism argument then shows ev_J has degree 2 for J regular. Finally, Gromov compactness implies surjectivity for any J .

We deduce that for any smooth almost complex structure J on M , there is a $u : (D^2, \partial D^2) \rightarrow (\mathbb{C}P^n, \mathbb{R}P^n)$ such that

$$\ell(u|_{\partial D^2}; g_J) \geq 2Diam(L; g_J) = 4\pi Diam(L; g_J) Area(u; g_J).$$

So, for any ω -tame J , we have $F_1(M, L, J) \geq 4\pi Diam(L; g_J)$, which implies

$$h_1(\omega) \geq 2\pi.$$

On the other hand, it follows from the discussion of Section 1.1 that $F_1(J_{st}, \omega) \leq 2\pi$ and $Diam(L; g_{J_{st}}) = \frac{1}{2}$, which implies that $h_1 \leq 2\pi$. Combining the two inequalities, we conclude $h_1(\omega) = 2\pi$. \square

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